Creating a database of rigorous Maass Forms

David Lowry-Duda
ICERM & Brown University

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This is about Maass forms in the LMFDB. You can look at them while I describe them and the work that went into their computation!

`https://www.lmfdb.org/ModularForm/GL2/Q/Maass/`

Before describing more, I should note that this database relies on the work of many others, and in particular the computations of Booker, Child, Lee, Seymour-Howell, Strömbergsson, and Venkatesh; and the previous heuristic database of Maass forms due largely to Stefan Lemurell and Fredrik Strömberg.

What is a Maass form?

Let $\Gamma_0(N) < \mathrm{SL}(2,\mathbb{Z})$ be a congruence subgroup. The Laplace-Beltrami operator acting on the upper halfplane $\mathcal{H}$ with the hyperbolic metric is given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

A Maass cuspform (of weight 0 and trivial nebentypus) on $\Gamma_0(N)$ are real-analytic eigenfunctions of Δ satisfying

1. $\Delta f = \lambda f$ (f has the eigenvalue λ)
2. $f(\gamma z) = f(z)$ (f is invariant under $\Gamma_0(N)$)
3. $f \in L^2(\Gamma_0(N)\backslash\mathcal{H})$ (f is a cuspform)

Why care about Maass forms?

Maass cuspforms form the discrete component of the spectral resolution of Δ. Stated differently, any $g \in L^2(\Gamma_0(N) \backslash H)$ will have an expansion of the form

$$g(z) = \sum_{\lambda \text{ eigenvalue}} \langle f_\lambda, g \rangle f_\lambda(z) + \sum_{\text{cusps}} (\text{Eisenstein series}).$$

Maass forms also have L-functions and are (a poorly understood) part of the Langlands program.

One major challenge is that all data associated to a generic Maass form is conjecturally transcendental, and conjecturally algebraically independent from each other and reasonable constants.

Today, we look at Maass newforms on $\Gamma_0(N)$ of weight 0. Each of these forms has an expansion

$$f_\lambda(z) = \sum_{n \geq 1} \frac{a(m)}{\sqrt{m}} W_\lambda(2\pi m y) \mathrm{cs}(2\pi m x),$$

where W_λ is a Whittaker function (a modified K-Bessel function of the third kind) and cs($\cdot$) is either cos($\cdot$) or sin($\cdot$), depending on the symmetry type of the Maass form.

By "rigorously compute a Maass form", we mean to rigorously estimate the eigenvalue λ and to rigorously estimate the coefficients $a(m)$. It is common to write the eigenvalue as $\lambda = \frac{1}{4} + r^2$ for some $r > 0$, called the spectral parameter. On the LMFDB, we give rigorous bounds for the spectral parameter and coefficients.

Looking at an example makes it clearer:

`https://www.lmfdb.org/ModularForm/GL2/Q/Maass/1.0.1.5.1`

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⌂ → Modular forms → Maass → Level 1 → Weight 0 → Character 1.1

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Maass form 1.5 on $\Gamma_0(1)$ with $R = 16.1380731$

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Maass form invariants

Level: 1

Weight: 0

Character: 1.1

Symmetry: odd

Fricke sign: $+1$

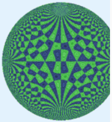
Spectral parameter: $16.1380731715210305801982942859860\ldots \pm 2 \cdot 10^{-91}$ (toggle for full precision)

Maass form coefficients

The coefficients here are shown to at most 8 digits of precision. Full precision coefficients are available in the downloads.

$a_1 = +1$	$a_2 = +1.16185559 \pm 1 \cdot 10^{-8}$	$a_3 = -1.28197256 \pm 1 \cdot 10^{-8}$
$a_4 = +0.34990842 \pm 1 \cdot 10^{-8}$	$a_5 = -0.75680641 \pm 1 \cdot 10^{-8}$	$a_6 = -1.48946699 \pm 1 \cdot 10^{-8}$
$a_7 = -0.29851912 \pm 1 \cdot 10^{-8}$	$a_8 = -0.75531254 \pm 1 \cdot 10^{-8}$	$a_9 = +0.64345365 \pm 1 \cdot 10^{-8}$
$a_{10} = -0.87929976 \pm 1 \cdot 10^{-8}$	$a_{11} = +0.76409070 \pm 1 \cdot 10^{-8}$	$a_{12} = -0.44857299 \pm 1 \cdot 10^{-8}$
$a_{13} = +0.16260232 \pm 1 \cdot 10^{-8}$	$a_{14} = -0.34683611 \pm 1 \cdot 10^{-8}$	$a_{15} = +0.97020506 \pm 1 \cdot 10^{-8}$
$a_{16} = -1.22747252 \pm 1 \cdot 10^{-8}$	$a_{17} = +0.41641743 \pm 1 \cdot 10^{-8}$	$a_{18} = +0.74760022 \pm 1 \cdot 10^{-8}$
$a_{19} = -0.75567638 \pm 1 \cdot 10^{-8}$	$a_{20} = -0.26481293 \pm 1 \cdot 10^{-8}$	$a_{21} = +0.38269332 \pm 1 \cdot 10^{-8}$

Properties



Label	1.5
Level	1
Weight	0
Character	1.1
Symmetry	odd
R	16.138
Fricke sign	$+1$

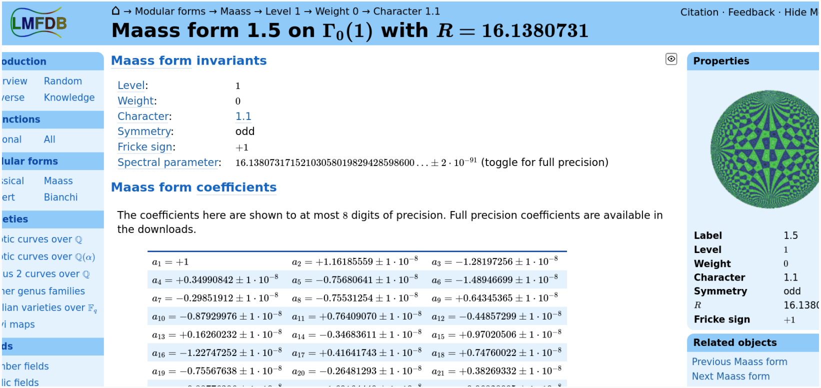
Related objects

Previous Maass form

Next Maass form

Looking at an example makes it clearer:

`https://www.lmfdb.org/ModularForm/GL2/Q/Maass/1.0.1.5.1`



Included data: level, weight, character, symmetry, Fricke sign, spectral parameter, coefficients, label, and a plot.

We've already described the level, weight, character, and spectral parameter.

Each newform is an eigenform of the Hecke operators T_p, the Atkin-Lehner involution ω_N, and the reflection operator T_{-1} sending $z \mapsto -\bar{z}$. The **Fricke sign** gives the eigenvalue under ω_N, and the **symmetry** is even if $T_{-1}f = f$ and is odd if $T_{-1}f = -f$.

The coefficients encode the Hecke eigenvalues (as with holomorphic modular forms or elliptic curve L-functions).

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(In practice, we always compute the symmetry type and a good bound for the spectral parameter. We only sometimes compute the coefficients to meaningful accuracy. When this accuracy is very poor [$\approx$ 40 percent of the time], we sometimes didn't compute the Fricke sign rigorously. In forthcoming work I intend to add all missing rigorous Fricke signs.)

The **label** has format $N.k.a.m.d$, where

1. N is the level
2. k is the weight
3. $N.a$ is the Conrey label of the nebentypus character
4. m is the index of the spectral parameter among the ordered list of spectral parameters of Maass forms with that level, weight, and character
5. d is a reserved disambiguation index in case there are multiple forms with the same eigenvalue.

For the *current* database, all forms have weight 0, trivial character, and $d = 1$. Thus we also have a **short label** $N.m$ that is shown.

This data was computed using a combination of methods.

For level 1, we used *quasimode construction* data due to Child [Chi22]. In this method, one starts with a heuristic Maass form $\widetilde{f}_\lambda$. One can show that if $(\Delta - \lambda)\widetilde{f}_\lambda$ has small L^2 norm, then $\widetilde{f}_\lambda$ is close to a true eigenfunction.

One can then use a trace formula to verify that every Maass form with eigenvalue up to some bound was found.

For higher level, the key ingredient was a rigorous implementation of the Selberg Trace Formula due to Seymour-Howell [SH22]. To overly simplify, this allows one to compute rigorously expressions of the form

$$\sum_{\lambda} F(\lambda) a_{\lambda}(n),$$

where $F(\cdot)$ is a "nice" test function and $a_{\lambda}(n)$ are the coefficients of the Maass form with eigenvalue λ. Through careful combinations of different test functions, it's possible to isolate individual eigenvalues and coefficients. As with level 1, this also allows one to guarantee that all Maass forms have been found in a given eigenvalue range.

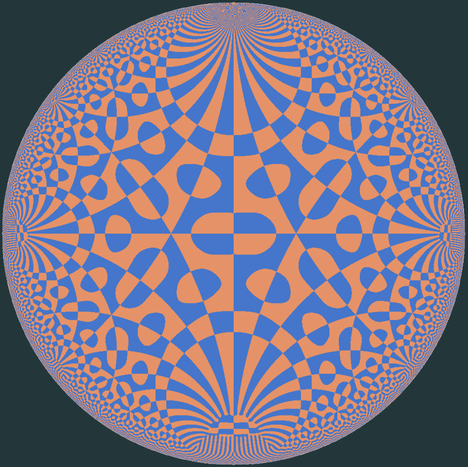
If the eigenvalue is computed to sufficient accuracy, it is then possible to use a rigorous version of Hejhal's algorithm [LDSH?] to refine and improve the bounds. Hejhal's algorithm uses truncated expansions and automorphy to construct approximate linear systems for the coefficients.

(I'd be happy to describe any of these methods in further detail in person.)

Forthcoming work

Currently about 40% of the forms in the database don't have a rigorous Fricke eigenvalue. In the near future, I'm working to make many (most?) of these rigorous.

Much more broadly — Bober, Booker, Knightly, Krishnamurthy, Lee, Lowry-Duda, and Seymour-Howell are working to construct and implement a generalized trace formula that will allow nontrivial nebentypus and general weight. This would allow a significant expansion of the current database (and would make full use of the label).



Thank you.

These slides are (or will be) on my website.

[Chi22]: Kieran Child. Twist-minimal trace formula for holomorphic cusp forms. Res. Number Theory, 8(1):Paper No. 11, 27, 2022

[LDSH?]: David Lowry-Duda and Andrei Seymour-Howell. A rigorous implementation of Hejhal's algorithm. 202?

[SH22]: Andrei Seymour-Howell. Rigorous computation of Maass cusp forms of squarefree level. Research in Number Theory, 8(4):83, 2022.