Surprising depth to the Fibonacci zeta function

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The Fibonacci zeta function

Fibonacci numbers and their zeta functions

Let $F(n)$ denote the nth Fibonacci number, defined through the linear recurrence $F(n+2) = F(n+1) + F(n)$ with initial conditions $F(0) = 0, F(1) = 1$. As is surely familiar, the sequence begins

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

The full Fibonacci zeta function is the lacunary zeta function

$$\zeta_{Fib}(s) := \sum_{n \geq 1} \frac{1}{F(n)^s} = \frac{1}{1^s} + \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots$$

The Fibonacci numbers $F(n)$ grow exponentially, and thus it's trivial to see that the series converges for $\operatorname{Re} s > 0$.

We will also investigate the zeta function associated to odd-indexed Fibonacci numbers,

$$\Phi(s) := \sum_{n \geq 1} \frac{1}{F(2n-1)^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{5^s} + \frac{1}{13^s} + \cdots$$

Simple analytic continuation

Recall the classical formula

$$F(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Using binomial series, it is straightforward to give a meromorphic continuation for the Fibonacci zeta function and $\Phi(s)$. We find that

$$\begin{aligned}
\Phi(s) &= \sum_{n\geq 1} \frac{5^{s/2}}{(\phi^{2n-1}+\phi^{1-2n})^s} = 5^{s/2} \sum_{n=1}^{\infty} \phi^{(2n-1)s} \left(\phi^{4n-2}+1\right)^{-s} \\
&= 5^{s/2} \sum_{n=1}^{\infty} \phi^{(2n-1)s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\phi^{4n-2}\right)^{-s-k} \\
&= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} \frac{\phi^{s+2k}}{\phi^{2s+4k}-1},
\end{aligned}$$

which gives meromorphic continuation to $\mathbb{C}$.

What does this say about the analytic behavior?

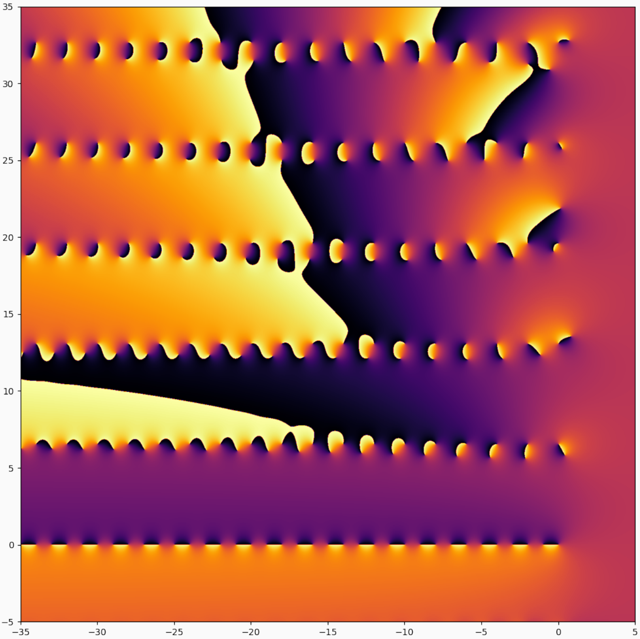
$$\Phi(s) = 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} \frac{\phi^{s+2k}}{\phi^{2s+4k} - 1}.$$

There are poles when $\phi^{2s+4k} - 1 = 0$, giving poles at

$$s = -2k + \ell \frac{\pi i}{\log \phi} \qquad (k \geq 0, \ell \in \mathbb{Z}).$$

There is a half-lattice of poles.

For fun, let's look at it.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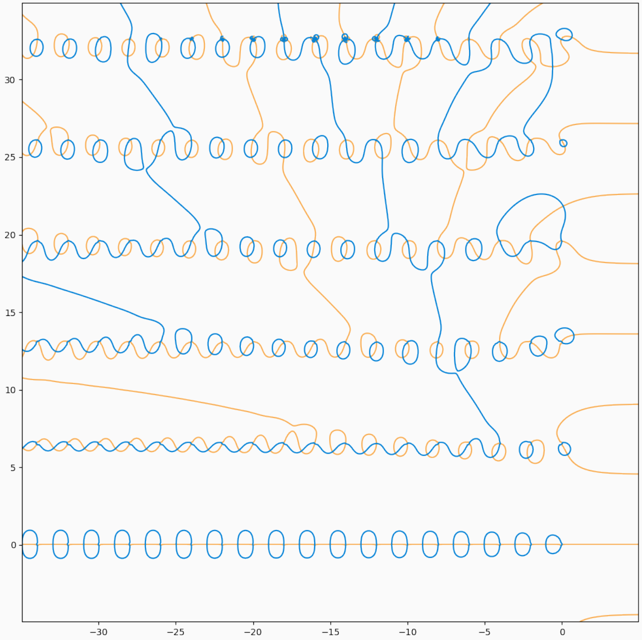

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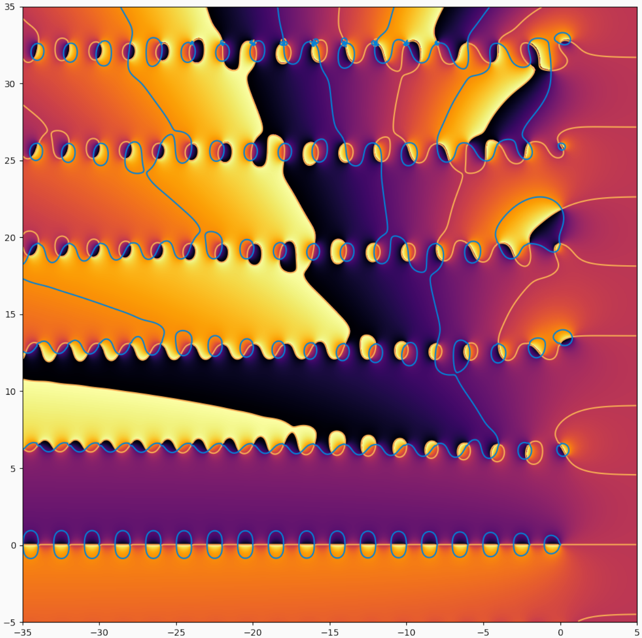
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Every pole comes close to a zero because $\Phi(s)$ grows too slowly for interesting behavior. Recall Jensen's Formula from early harmonic analysis.

For f (nontrivial) meromorphic on the closed disk $B_R(0)$, let $a_1, \dots a_p$ denote the zeros of f in B_R and let $b_1, \dots, b_q$ denote the poles of f in B_R (counting multiplicity for both). Write $f = c_f z^{\mathrm{ord}(0)} + \dots$, where c_f is the leading coefficient of the Laurent expansion at 0. Then

$$\log|c_f| = \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{i=1}^{p} \log\left|\frac{R}{a_i}\right| + \sum_{j=1}^{q} \log\left|\frac{R}{b_j}\right| - (\mathrm{ord}(0)) \log R.$$

In this case, ignoring constants and estimating growth for $\Phi(s)$, we find that

$$-\sum_{i=1}^{p} \log\left|\frac{R}{a_i}\right| + \sum_{j=1}^{q} \log\left|\frac{R}{b_j}\right| \ll \log R.$$

Suggesting a modular connection

These zeta functions exist in the literature. Landau (inconclusively) studied the value $\zeta_{Fib}(1)$ in [Lan99], but noted that $\Phi(1)$ can be expressed as special values of classical theta functions:

$$\Phi(1) = \frac{\sqrt{5}}{4}\theta_2^2\left(\frac{3-\sqrt{5}}{2}\right),$$

where

$$\theta_2(q) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}.$$

And there are a series of more recent results showing that $\zeta_{Fib}(2k)$ is transcendental for all $k \geq 1$ (analogous to $\zeta(2k)$) (due to Duverney, Nishioka, Nishioka, Shiokawa, Nesterenko, and others).

The idea is to combinatorially represent these special values as a nontrivial polynomial of certain Eisenstein series, and then to use a general theorem of Nesterenko on transcendentality of Eisenstein series.

Connections to modular forms

Let $r_1(n) = \#\{n = m^2 : m \in \mathbb{Z}\}$ (essentially a square-indicator function). Then the classical theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = \sum_{n \geq 0} r_1(n) e^{2\pi i n z}$$

is a (weight $1/2$) modular form on $\Gamma_0(4)$, and its coefficients recognize squares.

Our key fact for relating the Fibonacci numbers to modular forms is the following criterion for determining whether a number N is Fibonacci.

Lemma

A nonnegative integer N is a Fibonacci number iff either $5N^2 + 4$ or $5N^2 - 4$ is a square. Further, N is an odd-indexed Fibonacci number iff $5N^2 - 4$ is a square, and even-indexed iff $5N^2 + 4$ is a square.

(We'll return to this lemma later).

Shifted convolutions

n is an odd-indexed Fibonacci number iff $5n^2 - 4$ is a square. In terms of r_1, this is equivalent to requiring that

$$r_1(5n^2-4) \neq 0 \iff r_1(5n-4)r_1(n) \neq 0.$$

Thus

$$\Phi(s) = \sum_{n\geq 1} \frac{1}{F(2n-1)^s} = \frac{1}{4}\sum_{n\geq 1} \frac{r_1(5n-4)r_1(n)}{n^{s/2}},$$

which is a shifted convolution Dirichlet series formed from θ. Given modular forms $f = \sum a(n)e(nz)$ and $g = \sum b(n)e(nz)$, there is a general procedure one might try to follow to understand shifted convolutions

$$\sum_{n\geq 1} \frac{a(n)b(n\pm h)}{n^s},$$

building on ideas of Selberg, Sarnak, Goldfeld, Hoffstein, Hulse (and others).

The idea here is to consider $V(z) = \theta(5z)\overline{\theta(z)}y^{1/2}$, which is a weight 0 automorphic form on $\Gamma_0(20, \chi)$ whose 4th Fourier coefficient is

$$\sqrt{y} \sum_{n \geq 1} r_1(5n-4) r_1(n) e^{-20n\pi y}.$$

To study this as a Dirichlet series, it is convenient to use the real analytic Poincaré series

$$P_4(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(20)} \operatorname{Im}(\gamma z)^s e^{2\pi i 4 \gamma z} \chi(\gamma).$$

Then one can compute that

$$4\Phi(2s) = \sum_{n \geq 1} \frac{r_1(m) r_1(5m-4)}{m^s} = \frac{(20\pi)^s \langle V, P_4(\cdot, \overline{s} + \frac{1}{2}) \rangle}{\Gamma(s)}.$$

Thus the (odd-indexed) Fibonacci zeta function $\Phi(s)$ can be recognized as an inner product between automorphic forms.

The Poincaré series $P_4(z, s)$ has meromorphic continuation to $\mathbb{C}$ and is in $L^2(\Gamma_0(20, \chi)\backslash\mathcal{H})$, and thus the equality

$$4\Phi(2s) = \frac{(20\pi)^s \langle V, P_4(\cdot, \overline{s} + \frac{1}{2})\rangle}{\Gamma(s)}$$

reproves an abstract meromorphic continuation of $\Phi(s)$.

But there is something unusual going on. The Poincaré series $P_4(z, s)$ has an enormous number of chaotic, transcendental, poorly understood poles.

One way to understand the meromorphic continuation of the $P_4(z, s)$ is to use its spectral decomposition.

$$P_4(z, s) = \sum_j \langle \mu_j, P_4(\cdot, s) \rangle \mu_j(z) + \text{(continuous)},$$

where μ_j ranges over a basis of Maass eigenforms and the 'continuous' part refers to a sum over Eisenstein series.

(It turns out that the first sets of poles all come from the discrete spectrum (except for a distinguished pole at $s = 0$), and we will focus entirely on the discrete spectrum in this talk).

We study analytic behavior of $\Phi(2s)$ through

$$4\Phi(2s) = \frac{(20\pi)^s}{\Gamma(s)} \sum_j \langle \mu_j, P_4(\cdot, s) \rangle \langle V, \mu_j \rangle + \text{(continuous)}.$$

But not every pole from the Poincaré series yields a pole in $\Phi(2s)$. Many parts of the spectral expansion disappear. In particular, the only Maass forms that contribute are self-dual.

Lemma

$\langle V, \mu_j \rangle = 0$ *unless* μ_j *is self-dual.*

(proof sketch).
Recognize θ as the residue of the weight 1/2, level 20 Eisenstein series $E^{\frac{1}{2}}(z, w)$. This will work as the space of modular forms of weight 1/2 on $\Gamma_0(20)$ is 1-dimensional. Then

$$\begin{aligned}\langle V, \mu_j \rangle &= \langle y^{\frac{1}{2}} \theta(5z)\overline{\theta(z)}, \mu_j \rangle = c \operatorname{Res}_{w=\frac{3}{4}} \langle y^{\frac{1}{4}} \theta(5z)\overline{E^{\frac{1}{2}}(z, w; \Gamma_0(20))}, \mu_j \rangle \\ &= c' \operatorname{Res}_{w=\frac{3}{4}} \frac{\Gamma(w - \frac{1}{4} + it_j)\Gamma(w - \frac{1}{4} - it_j)}{(10\pi)^w \Gamma(w + \frac{1}{4})} \sum_{n \geq 1} \frac{\rho_j(-5n^2)}{n^{2w - \frac{1}{2}}}.\end{aligned}$$

The inner product against the Eisenstein series leads to a Rankin-Selberg type expansion for what is nearly the symmetric square L-function associated to μ_j. □

Self-dual Maass forms

Self-dual forms of the type that contribute were studied by Maaß himself.

Let $\eta(\mathfrak{b})$ be the Hecke character on $\mathbb{Q}(\sqrt{5})$ by

$$\eta\big((a+b\sqrt{2})\big) = \operatorname{sgn}(a+b\sqrt{5})\operatorname{sgn}(a-b\sqrt{5})\Big|\frac{a+b\sqrt{5}}{a-b\sqrt{5}}\Big|^{\frac{i\pi}{2\log((1+\sqrt{5})/2)}}.$$

We note that the number $\phi = (1+\sqrt{5})/2$ is a fundamental unit for $\mathbb{Q}(\sqrt{5})$, and that defining η on principle ideals is sufficient as $\mathcal{O}(\sqrt{5})$ is a PID. For each integer m, consider the function

$$\mu_m(z) := \sum_{n\geq 1}\sum_{N(\mathfrak{b})=n} \eta(\mathfrak{b})^m \sqrt{y} K_{\frac{im\pi}{2\log((1+\sqrt{5})/2)}}(2\pi ny)\cdot \begin{cases} \cos(2\pi nx), & 2\nmid m \\ \sin(2\pi nx), & 2 \mid m. \end{cases}$$

Following Maaß, and as recounted in [Bum98, Theorem 1.9.1], the functions $\mu_m(z)$ are Maass cusp forms for $\Gamma_0(20)$ with nebentypus χ. The coefficients of μ_m are real, and thus self-dual.

These are dihedral Maass forms.

Polar comparison: binomial continuation

As a quick check, we examine the first line of poles. From the simple binomial expression, we have the continuation

$$4\Phi(2s) = 4 \cdot 5^s \sum_{k=0}^{\infty} \binom{-2s}{k} \frac{\phi^{2s+2k}}{\phi^{4s+4k} - 1},$$

so that the poles on the line $\operatorname{Re} s = 0$ all come from the single term $5^s \phi^{2s} / (\phi^{4s} + 1)$, which are at

$$s = \ell \frac{\pi i}{2 \log \phi} \qquad (\ell \in \mathbb{Z}).$$

Polar comparison: modular continuation

From the discrete portion of the continuation of $P_4(z, s)$, we have

$$
\begin{aligned}
4\Phi(2s) &\approx \frac{(20\pi)^s}{\Gamma(s)} \sum_j \langle \mu_j, P_4(\cdot, s)\rangle \langle V, \mu_j\rangle \\
&\approx \frac{(20\pi)^s}{\Gamma(s)} \sum_j \frac{\rho_j(4)\sqrt{\pi}\Gamma(s+it_j)\Gamma(s-it_j)}{(16\pi)^s\Gamma(s+\frac{1}{2})} \langle V, \mu_j\rangle,
\end{aligned}
$$

which has potential poles at $s = \pm i t_j$ along the line $\operatorname{Re} s = 0$. Here, t_j is the "type" associated to the Maass form. The "types" associated to the dihedral Maass forms above are exactly

$$
i t_m = \frac{m\pi i}{2\log\phi} \qquad (m \in \mathbb{Z}, m \neq 0).
$$

Thus the poles of $\Phi(2s)$ line up perfectly with the poles coming from the dihedral Maass forms (and a distinguished pole at $s = 0$). This story continues for all poles, not just those on $\operatorname{Re} s = 0$.

Rolling up your sleeves and explicitly compute every component of the spectral resolution gives an alternate, explicit meromorphic continuation:

$$\Phi(s) = \frac{5^{s/2}}{8\Gamma(s)\log\phi} \sum_{m\in\mathbb{Z}} (-1)^m \Gamma\Big(\frac{s}{2} + \frac{\pi i m}{2\log\phi}\Big)\Gamma\Big(\frac{s}{2} - \frac{\pi i m}{2\log\phi}\Big).$$

(See [AKLDW25] for details).

The growth properties of $\Phi(s)$ are clear in this expansion.

A binomial expansion of $\Phi(s)$ shows a half-lattice of poles. A spectral expansion shows poles coming from eigenvalues of lots of Maass forms, and the "only" way to get a half-lattice is if almost all Maass forms don't contribute.

This work suggests that $\Phi(s)$ should have an intrinsic expression as a dihedral Galois representation...but I don't know how to find it.

Generalized Fibonacci Zeta Functions

The key idea to the method of recognizing the relationship was the lemma relating Fibonacci numbers to squares.

Lemma

A nonnegative integer N is a Fibonacci number iff either $5N^2 + 4$ or $5N^2 - 4$ is a square. Further, N is an odd-indexed Fibonacci number iff $5N^2 - 4$ is a square, and even-indexed iff $5N^2 + 4$ is a square.

Explanation of Lemma

We can view this lemma as describing behavior of units in $\mathcal{O}(\sqrt{5})$. Any integer in $\mathcal{O}(\sqrt{5})$ can be written uniquely as

$$x = m + n\tfrac{5+\sqrt{5}}{2} \qquad \left(= \frac{u + n\sqrt{5}}{2} \text{ with } u = 2m + 5m \right)$$

and x is a unit iff $N(x) = \pm 1$, which is equivalent to the condition that

$$u^2 = 5n^2 \pm 4.$$

Suppose u and n are a positive solution making x a unit. As ϕ is a fundamental unit

$$x = \frac{u + n\sqrt{5}}{2} = \phi^r.$$

Recall Binet's formulas for the Fibonacci and Lucas numbers, which express the nth Lucas or Fibonacci numbers in terms of ϕ.

$$L(r) = \phi^r + \overline{\phi}^r, \qquad F(r) = \frac{\phi^r - \overline{\phi}^r}{\sqrt{5}}.$$

Then

$$\begin{aligned} x = \frac{u + n\sqrt{5}}{2} = \phi^r &= \frac{1}{2}\left[(\phi^r + \overline{\phi}^r) + \frac{\phi^r - \overline{\phi}^r}{\sqrt{5}}\sqrt{5}\right] \\ &= \frac{1}{2}\left[L(r) + F(r)\sqrt{5}\right], \end{aligned}$$

where $L(r)$ are the Lucas numbers and $F(r)$ are the Fibonacci numbers.

Explanation of Lemma (cont.)

Thus if there is a (positive) solution (u, n) to $u^2 = 5n^2 \pm 4$, then

$$\frac{u + n\sqrt{5}}{2} = \frac{L(r) + F(r)\sqrt{5}}{2}$$

for some r, and thus n is Fibonacci. Conversely, if $n = F(r)$ for some r, then

$$\phi^r = \frac{1}{2}\Big[L(r) + F(r)\sqrt{5}\Big] \implies L(r)^2 - 5F(r)^2 = \pm 4,$$

and thus n is part of a solution to $u^2 = 5n^2 \pm 4$.

The condition that $5n^2 \pm 4$ is a square is really an indicator that a particular element is a unit in a ring of integers. This generalizes readily.

Generalization of Lemma

We can generalize this lemma to describe the behavior of units in $\mathcal{O}(\sqrt{d})$. Any integer in $\mathcal{O}(\sqrt{d})$ can be written uniquely as

$$x = m + n\tfrac{q+\sqrt{q}}{2}, \qquad \begin{cases} q = d & d \equiv 1 \bmod 4 \\ q = 4d & d \equiv 2, 3 \bmod 4 \end{cases}$$

and x is a unit iff $N(x) = \pm 1$, which is equivalent to the condition that

$$u^2 = qn^2 \pm 4, \qquad (\text{where } u = 2m + qn).$$

Suppose u and n are a positive solution making x a unit. Let ε be a fundamental unit

$$\begin{aligned} x = \frac{u + n\sqrt{q}}{2} = \varepsilon^r &= \frac{1}{2}\left[(\varepsilon^r + \bar{\varepsilon}^r) + \frac{\varepsilon^r - \bar{\varepsilon}^r}{\sqrt{q}}\sqrt{q}\right] \\ &= \frac{1}{2}\left[L_{\sqrt{d}}(r) + F_{\sqrt{d}}(r)\sqrt{q}\right], \end{aligned}$$

where $L_{\sqrt{d}}(r) = \mathrm{Tr}(\varepsilon^r)$ are $\sqrt{d}$-Lucas numbers and $F_{\sqrt{d}}(r) = \mathrm{Tr}(\varepsilon^r/\sqrt{q})$ are $\sqrt{d}$-Fibonacci numbers.

Generalization of Lemma (cont.)

Thus if there is a (positive) solution (u, n) to $u^2 = qn^2 \pm 4$, then

$$\frac{u + n\sqrt{q}}{2} = \frac{L_{\sqrt{d}}(r) + F_{\sqrt{d}}(r)\sqrt{q}}{2}$$

for some r, and thus n is $\sqrt{d}$-Fibonacci. Conversely, if $n = F_{\sqrt{d}}(r)$ for some r, then

$$\varepsilon^r = \frac{1}{2}\left[L_{\sqrt{d}}(r) + F_{\sqrt{d}}(r)\sqrt{q}\right] \implies L_{\sqrt{d}}(r)^2 - qF_{\sqrt{d}}(r)^2 = \pm 4,$$

and thus n is part of a solution to $u^2 = qn^2 \pm 4$.

(Note that if $d = 2$ or $d \equiv 3 \bmod 4$, the equality $q = 4d$ has the effect of making most 4s appearing above to factor out).

From this point of view, the major idea is that the Fibonacci numbers $F(n)$ are traces of powers of the fundamental unit (divided by $\sqrt{5}$),

$$F(n) = \mathrm{Tr}(\phi^n/\sqrt{5}).$$

For the ring of integers associated to a quadratic extension $\mathbb{Q}(\sqrt{d})$, if we define $\sqrt{d}$-Fibonacci numbers as

$$F_{\sqrt{d}}(n) = \mathrm{Tr}(\varepsilon^n/\sqrt{q})$$

as above, then the lemma applies and $\sqrt{d}$-Fibonacci numbers and we see that $\sqrt{d}$-Fibonacci numbers are detectable via a quadratic form that can be built out of theta functions.

Trace zeta function approach

If the fundamental unit ε satisfies $N(\varepsilon) = -1$, then the proof methods described above for $\Phi(s)$ show that

$$\Phi_d(s) = \sum_{n \geq 1} \frac{1}{F_{\sqrt{d}}(2n-1)^s} = \sum_{n \geq 1} \frac{1}{\mathrm{Tr}(\varepsilon^{2n-1}/\sqrt{q})^s},$$

using $V_d = \theta(qz)\overline{\theta(z)}$ in place of V. It also remains true that the poles come from self-dual Maass forms.

(If there are not units of norm -1, then one must instead study the series

$$\sum_{n \geq 1} \frac{r_1(qn+4)r_1(n)}{n^s},$$

with a $+$ instead of a $-$. For technical reasons, it is necessary to perform a different continuation of this series. I don't get into that in this talk).

Pell's Equation

The equations $u^2 = qn^2 \pm 4$ are Pell equations. It is also possible to construct a zeta function by interpreting the Pell equation directly as a quadratic form.

For example, we will consider the Pell equations

$$x^2 - 2y^2 = -h, \qquad (h \in \mathbb{N}_{>0}).$$

Solutions do not exist for every h, but when solutions exist they are exponentially sparse and satisfy a linear recurrence relation.

Analogous with the Fibonacci-zeta case, we can recognize this zeta function as

$$4D_h(s) = \sum_{m \geq 1} \frac{r_1(m) r_1(2m - h)}{m^s}.$$

(The identification to solutions $x^2 - 2y^2 = -h$ is through $y^2 = m$).

Alternately, we note that for each h there exists a number $d = d(h)$ of *fundamental* solutions $(u_1, v_1), \ldots, (u_d, v_d)$. Then the y part of the solutions are given by the linear recurrences

$$y_k(n) = 6y_k(n-1) - y_k(n-2) = \alpha_k(3+2\sqrt{2})^n + \beta_k(3-2\sqrt{2})^n,$$

where $\alpha_k = \frac{1}{2}v_k + \frac{1}{2\sqrt{2}}u_k$ and $\beta_k = \frac{1}{2}v_k - \frac{1}{2\sqrt{2}}u_k$. The exact fundamental solutions are not trivial to determine in general.

For any fixed h, it is straightforward to adapt the binomial series method to provide an analytic continuation for the lacunary Dirichlet series formed from the solutions $y_k(n)$. Let $\omega = 3 + 2\sqrt{2}$, and note that $\omega^{-1} = \overline{\omega}$. Then we define

$$D_h(s) = \sum_{\substack{n \geq 0 \\ k \leq d}} \frac{1}{(\alpha_k \omega^n + \beta_k \omega^{-n})^{2s}} = \sum_{k \leq d} \frac{1}{\alpha_k^{2s}} \sum_{n \geq 0} \frac{\omega^{-2ns}}{(1 + (\beta_k/\alpha_k)\omega^{-2n})^{2s}}.$$

This latter expression has meromorphic continuation to the plane and is analytic for $\operatorname{Re} s > 0$ (via binomial expansion).

To compare with the previous trace-zeta function, note that $\varepsilon = 1 + \sqrt{2}$ is a fundamental unit, $N(\varepsilon) = -1$, and $\omega = \varepsilon^2 = 3 + 2\sqrt{2}$.

Thus the linear recurrences defining the solutions $y_k(n)$ are in terms of ε^2 and $\overline{\varepsilon^2}$. The major distinction is that the initial conditions for the linear recurrences are different (and there may be multiple, depending on h in $x^2 - 2y^2 = -h$).

We recognize this again as a shifted convolution, now with $V = \theta(2z)\overline{\theta(z)}\sqrt{y}$, which is a modular form on $\Gamma_0(8, \chi)$. The hth Fourier coefficient of V contains the relevant arithmetic data, and we use a Poincaré series that extracts the hth Fourier coefficient:

$$P_h(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(8)} \mathrm{Im}(\gamma z)^s e^{2\pi i h \gamma z} \chi(\gamma).$$

Then one can compute that

$$D_h(s) = \frac{(8\pi)^s \langle V, P_h(\cdot, \overline{s} + \frac{1}{2}) \rangle}{\Gamma(s)},$$

and abstractly we get another continuation.

This same construction applies to any Pell equation (and restricting to Pell equations of the for $x^2 - dy^2 = -h$ (with a minus) accomplishes the same minor technical detail as requiring $N(\varepsilon) = -1$ previously).

Remarks on $x^2 - 2y^2 = -h$ case

In unpublished work, I examined the $x^2 - 2y^2 = -h$ in detail.

Theorem

For $h \geq 1$ and $\operatorname{Re} s \gg 1$, we have that

$$D_h(s) = \frac{2^s \sqrt{\pi} \sigma_0^\chi(h) \Gamma(s)}{\log(1+\sqrt{2}) h^s \Gamma(s+\frac{1}{2})} + 2^s \sqrt{\pi} \sum_j \frac{\rho_j(h)}{h^s} \frac{G(s+\frac{1}{2}, it_j)}{\Gamma(s)} \langle V, \mu_j \rangle$$

in which $G(s,z) = \Gamma(s - \frac{1}{2} + z)\Gamma(s - \frac{1}{2} - z)/\Gamma(s)$.

Here, $\sigma_0^\chi(h) = \sum_{d|h} \chi(d)$, the first term comes exactly from a dihedral Eisenstein series and all the Maass forms that appear are again dihedral.

Remarks (continued)

Comparing residues of poles at $s = 0$ across the binomial representation and the modular representation shows that

$$\frac{2d}{\log \omega} = \mathrm{Res}_{s=0} \frac{2^s \sqrt{\pi} \sigma_0^\chi(h) \Gamma(s)}{\log(1+\sqrt{2}) h^s \Gamma(s+\frac{1}{2})} = \frac{\sigma_0^\chi(h)}{\log(1+\sqrt{2})},$$

where $d = d(h)$ is the number of *fundamental* solutions to the Pell equation.

Recalling that $\omega = (1+\sqrt{2})^2$, we see that that $d = \sigma_0^\chi(h)$, which gives a class number formula for solutions to the Pell equation.[1]

[1] This is not a new result, but it is a nice result.

Relation to 3APs of Squares

The function $r_1(m)r_1(2m-h)$ trivially detects a 2AP of squares, $\{m, 2m-h\}$. Thus

$$4D_h(s) = \sum_{m \geq 1} \frac{r_1(m)r_1(2m-h)}{m^s}$$

as a Dirichlet series detects 2APs of squares as m ranges.

Naive question

Can we understand 3APs of squares $\{h, m, 2m-h\}$ by studying the (multiple) Dirichlet series

$$D(s,w) = \sum_{h \geq 1} \frac{4D_h(s)r_1(h)}{h^w} = \sum_{m,h \geq 1} \frac{r_1(h)r_1(m)r_1(2m-h)}{m^s h^w}?$$

This is a Dirichlet series formed from individual Pell-type Dirichlet series. Is it understandable?

Answer: Yes.

But not through the raw binomial series continuation of $D_h(s)$. The uncertain behavior of the fundamental solutions makes computing with the explicit binomial series untenable.

But the modular form continuation *is robust enough* to make sense of $D(s, w)$.

Heuristic explanation

From the evaluation

$$D_h(s) = \frac{2^s\sqrt{\pi}\sigma_0^\chi(h)\Gamma(s)}{\log(1+\sqrt{2})h^s\Gamma(s+\frac{1}{2})} + 2^s\sqrt{\pi}\sum_j \frac{\rho_j(h)}{h^s}\frac{G(s+\frac{1}{2}, it_j)}{\Gamma(s)}\langle V, \mu_j\rangle,$$

we can study what would come from $\sum_{h\geq 1} D_h(s) r_1(h) h^{-w}$. In the first term, the sums over h and j become

$$\sum_{h\geq 1}\frac{\sigma_0^\chi(h) r_1(h)}{h^{s+w}} = \sum_{h\geq 1}\frac{\sigma_0^\chi(h^2)}{h^{2s+2w}} \qquad \sum_{h\geq 1}\frac{\rho_j(h^2)}{h^{2s+2w}}.$$

Both of these are essentially symmetric square L-functions associated to well-studied objects, and are thus understandable.

This is enough to give meromorphic continuation in w, and $D(s,w)$ has meromorphic continuation to all of $\mathbb{C}^2$ [HKLDW20a].

Application: equidistribution

If a^2, b^2, c^2 is a 3AP of squares with $c^2 - b^2 = b^2 - a^2$, then $a^2 + c^2 = 2b^2$. Thus $(a/b, c/b)$ is a rational point on the circle $X^2 + Y^2 = 2$.

Let $A(b)$ denote the number of rational points on $X^2 + Y^2 = 2$ with (reduced) denominator b. Then

$$\sum_{d|b} A(d) = \#\{(a, c) \in \mathbb{Z}^2 : a^2 + c^2 = 2b^2\} = r_2(b^2).$$

As $r_2(n)/4$ is multiplicative, we can compute that

$$\sum_{n \geq 1} \frac{A(n)}{n^s} = \frac{4\zeta(s)L(s, \chi_4)}{(1 + 2^{-s})\zeta(2s)},$$

and Perron-type analysis shows that the number of (positive) rational points on $X^2 + Y^2 = 2$ of the reduced form $(a/b, c/b)$ with $1 \leq b \leq \sqrt{X}$ is

$$\sum_{b \leq X} = \frac{4}{\pi} X + O(X^{\frac{2}{3} + \epsilon}).$$

Equidistribution (cont)

The number of (positive) rational points on $X^2 + Y^2 = 2$ of the reduced form $(a/b, c/b)$ with $1 \leq b \leq \sqrt{X}$ is

$$\sum_{b \leq X} = \frac{4}{\pi} X + O(X^{\frac{2}{3}+\epsilon}).$$

If these rational points equidistribute, then the number of points with $a/b \leq \delta$ should be approximately

$$\frac{\arcsin(\delta/2)}{2\pi} \frac{4}{\pi} X.$$

Theorem (HKLDW)

These points do equidistribute, and satisfies the asymptotic

$$\frac{\arcsin(\delta/2)}{2\pi} \frac{4}{\pi} X + O(X^{\frac{3}{4}+\epsilon}).$$

Other applications

Theorem

The number of primitive 3APs of squares a^2, b^2, c^2 with $c^2 \leq X$ is

$$\frac{\sqrt{2}}{\pi^2}\log(1+\sqrt{2})\sqrt{X} + O(X^{\frac{3}{8}+\epsilon}).$$

Theorem

$(\cdots)$ with with $a^2 \leq Y$ and $b^2 \leq X$ is

$$\frac{1}{\sqrt{2}\,\pi^2}\sqrt{Y}\log(X/Y) + c\sqrt{Y} + O(X^{\epsilon}Y^{\frac{3}{8}+\epsilon}).$$

Theorem

$(\cdots)$ with $ab \leq X$ is

$$\frac{2\sqrt{2}}{\pi^2}\,{}_2F_1(\tfrac{1}{4},\tfrac{1}{2},\tfrac{5}{4},\tfrac{1}{2})\sqrt{X} + O(X^{\frac{3}{4}+\epsilon}).$$

Due to the connections between 3APs of squares, right triangles, and elliptic curves, we get several refined distributional results on right triangles. For example:

Theorem

The number of primitive integer right triangles with hypotenuse at most X and whose acute angles are within ω of $\pi/4$ is

$$\frac{2\omega}{\pi^2}X + O(X^{\frac{3}{4}+\epsilon}).$$

(Yes, I think this is funny-looking — but I'm thrilled that this comes from studying a multiple Dirichlet series of $\sqrt{2}$-Fibonacci zeta functions)!

Proof sketches

All of these are proved in essentially the same way — classic complex analytic number theory once we have a Dirichlet series with understandable meromorphic continuation.

For example, with

$$D(s, w) = \sum_{\substack{m,h \geq 1 \\ (m,h)=1}} \frac{r_1(h) r_1(m) r_1(2m-h)}{m^s h^w},$$

we have

$$\sum_{m \leq X} \sum_{h/m \leq \delta} r_1(m) r_1(h) r_1(2m-h) \approx \iint D(s-w, w) \frac{X^s}{s} \frac{\delta^w}{w} ds \, dw.$$

History of this project

In [HKLDW20b], we examined a naive shifted sum for detecting if a given number t is congruent:

$$\sum_{m,n \leq X} r_1(m+h)r_1(m-h)r_1(m)r_1(tn).$$

This sum is asymptotically of size $\sqrt{X}$ if t is congruent, and is otherwise 0. Thus whether t is congruent is determined by poles of

$$\sum_{m,n \geq 1} \frac{r_1(m+h)r_1(m-h)r_1(m)r_1(th)}{m^s h^w}.$$

This seemed like a potential refinement of Tunnell's theorem, but we were unable to understand this series. By counting congruent numbers (instead of detecting them), we arrive at $D(s, w)$, which we understand through $D_h(s)$ as above.

Thank you very much.

Please note that these slides (and references for the cited works) are (or will soon be) available on my website (davidlowryduda.com).

References i

Eran Assaf, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
The fibonacci zeta function and modular forms.
http://arxiv.org/abs/2502.01415v1, 2025.
arXiv:math.NT:2502.01415v1.

D. Bump.
Automorphic Forms and Representations, volume 55 of Cambridge Studies in Advanced Mathematics.
Cambridge University Press, 1998.

Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
Arithmetic progressions of squares and multiple Dirichlet series.
Math. Zeit., 2020.
https://arxiv.org/abs/2007.14324.

Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
A shifted sum for the congruent number problem.
Ramanujan J., 51(2):267–274, 2020.

E. Landau.
Sur la série des inverse de nombres de fibonacci.
Bull. Soc. Math. France, 27:298–300, 1899.

M. Ram Murty.
The Fibonacci zeta function.
In *Automorphic representations and L-functions. Proceedings of the international colloquium, Mumbai, India, January 3–11, 2012*, pages 409–425. New Delhi: Hindustan Book Agency; Mumbai: Tata Institute of Fundamental Research, 2013.