Rigorous Maass Forms

David Lowry-Duda

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The geometry of a space influences the functions that live on that space. For a motivating example, consider drumheads.

The frequencies and waves that a drumhead can make are determined by the Helmholtz equation,

$$\begin{cases} \Delta u + \lambda u = 0, \\ u_{\partial D} = 0. \end{cases}$$

Different drums admit different solutions, leading to different sounds.

Can one can hear the shape of a drum?

Answer: no!

I've been looking at the analogous problem on Fuchsian groups. Let Γ be a congruence subgroup in $\mathrm{SL}(2, \mathbb{Z})$. We call f a *Maass cuspform* with eigenvalue λ if

$$\begin{cases} \Delta f - \lambda f = 0, \\ f(\gamma z) = f(z), \\ f \in L^2(\Gamma \backslash \mathcal{H}). \end{cases}$$

Maass showed that for any fixed λ, there are at most finitely many solutions, but he was unsuccessful in finding any outside of very special, very particular constructions (via Hecke characters).

Can one hear the shape of a congruence subgroup?

Answer: Maybe?

We think all the data associated to a generic Maass form is transcendental. Maybe we'll never *exactly* compute a Maass form.

By "compute a Maass form", we mean to rigorously estimate the eigenvalue λ and to rigorously estimate the coefficients $a(m)$.

Each Maass form discussed today has an expansion

$$f(z) = \sum_{n \geq 1} \frac{a(m)}{\sqrt{m}} K_{ir}(2\pi my)\sqrt{y}\,\mathrm{cs}(2\pi mx),$$

where $\mathrm{cs}(\cdot)$ is either $\cos(\cdot)$ or $\sin(\cdot)$, depending on the symmetry type of the Maass form. This is a (real) analytic function on $\Gamma_0(N)\backslash\mathcal{H}$ for some squarefree N, and it is an eigenfunction of a Laplacian with eigenvalue $\lambda = \frac{1}{4} + r^2$.

The Past

Before last year, there was no database of rigorously computed Maass forms. But there were several efforts to heuristically approximate Maass forms, notably by Kuznetsov, Hejhal, Stark, Strömberg, Lemurell, Then, Farmer and their collaborators.

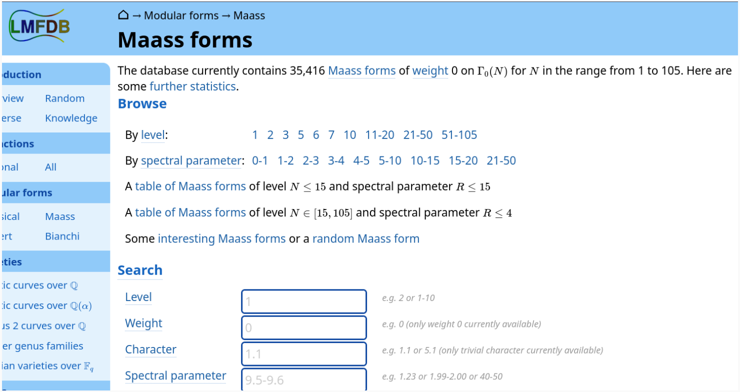
The LMFDB had a database of 14495 *heuristically estimated* Maass forms.[1]

[1] I'd be happy to talk about heuristic methods of computing Maass forms, but I don't discuss this today.

The Present

As of last year, the LMFDB now has a database of 35416 *rigorously approximated* Maass forms, combining code and efforts of Child, Seymour-Howell, and Lowry-Duda.

You can explore it now!
`https://www.lmfdb.org/ModularForm/GL2/Q/Maass/`

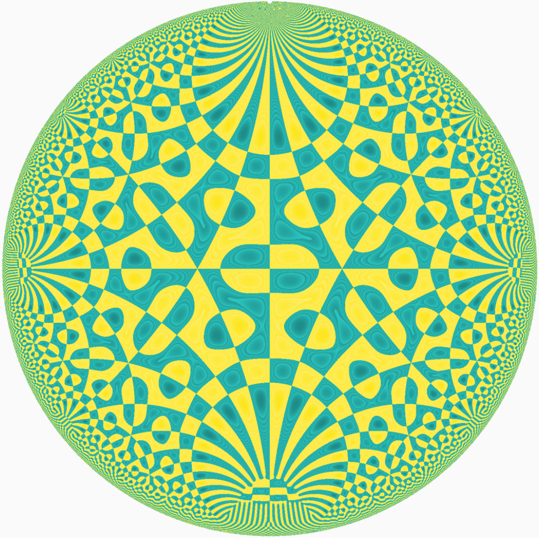


The database is the result of a combination of several algorithms.

1. For level 1 (consisting of 2202 Maass forms), we use an algorithm based on work of Booker, Strömbergsson, and Venkatesh. This is a *certification algorithm*, which certifies that a putative form is actually close to a true form.
2. For other levels, we first use a Selberg trace formula to compute rigorous (but sometimes weak) approximations, and then try to refine these with a rigorous form of Hejhal's algorithm.

One enormous benefit of starting with a trace formula is that we can guarantee that we've computed <u>every Maass form</u> for a fixed congruence subgroup up to a certain eigenvalue bound.

We can also look at what the *vibrations* of the modular surface look like.



The Future

There is much left to do.

1. With Bober, Booker, Knightley, Lee, and Seymour-Howell, we're working on developing new (explicit, implementable) trace formulas that would let us generalize to new levels, inner characters, and weights.
2. With Bieri, Butbaia, Costa, Deines, Lee, Oliver, Qi, and Veenstra, we're studying Maass form data using machine learning to heuristically help fill in gaps.[2] I'm currently using insights from this to *rigorously* add missing data.
3. And with Seymour-Howell, I'm working on using a version of the Kuznetsov trace formulas to compute rigorous, low-precision approximations.

[2] *Learning Fricke signs from Maass form coefficients*, arXiv: 2501.02105.

Thank you very much.

Please note that these slides will soon be) available on my website (davidlowryduda.com).