On Murmurations and Trace Formulas

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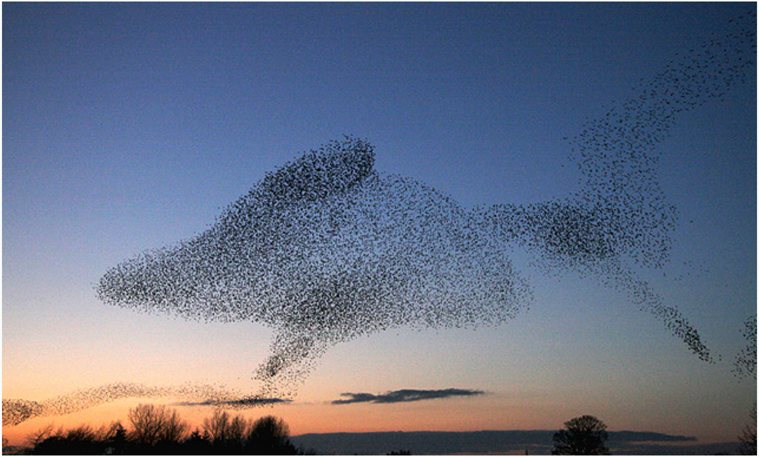


Starling shapes in the evening sky, Walter Baxter,
`https://www.geograph.org.uk/photo/1065181`.

The language of murmurations is based in L-functions.

A general (self-dual) L-function has the shape

$$L(s, \pi) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

and satisfies a functional equation

$$\Lambda(s, \pi) := N^s G(s) L(s, f) = \epsilon N^{1-s} G(1-s) L(1-s, f).$$

For functions associated to π in some family $\mathcal{F}$ (ordered by some height function, typically analytic conductor) murmuration phenomena are correlations between the coefficients $a(n)$ and the root numbers ϵ.

Given a smooth nonnegative weight function $\Phi : (0, \infty) \longrightarrow \mathbb{R}$ of compact support and a complex-valued function f defined on a family $\mathcal{F}$ of L-functions ordered with respect to a height function h, we look at

$$A_\Phi^{\mathcal{F}}(f, X) = A(f, X) := \sum_{\pi \in \mathcal{F}} \Phi(h(\pi)/X) f(\pi),$$

and more generally expected values

$$\mathop{\mathbb{E}}_{\pi \in \mathcal{F}}[f; X] := \frac{A(f, X)}{A(1, X)}.$$

For example, take $\mathcal{F} = \mathcal{E}^\pm$, elliptic curves ordered by conductor with root number $\epsilon = \pm 1$; and take Φ to be the indicator function on $[X, 2X]$. The well-known initial murmurations of elliptic curves are plots of

$$\mathbb{E}_{\mathcal{E}^\pm}[a_E(p)\sqrt{p}; X] = \frac{\sum_{\substack{E \in \mathcal{E}^\pm \\ X \leq \mathrm{cond}(E) \leq 2X}} a_E(p)\sqrt{p}}{\sum_{\substack{E \in \mathcal{E}^\pm \\ X \leq \mathrm{cond}(E) \leq 2X}} 1}$$

for various primes p and in various dyadic ranges $[X, 2X]$.

One Level Density

For "nice" families $\mathcal{F}$ of L-functions $L(s, f)$, ordered by conductor N_f, let $\mathcal{F}(N) = \{f \in \mathcal{F} : N_f = N\}$.

Katz and Sarnak predict that for large N, the low-lying zeros of $L(s, f)$ for $f \in \mathcal{F}(N)$ act like eigenvalues of matrices drawn randomly from certain groups of matrices associated to $\mathcal{F}$. One measure of zero behavior is one level density, which is

$$\mathrm{OLD}_\Phi(\mathcal{F}) := \lim_{N\to\infty} \frac{1}{\#\mathcal{F}(N)} \sum_{f\in\mathcal{F}(N)} \sum_{\gamma_f} \Phi\left(\frac{\gamma_f \log N}{2\pi}\right),$$

where γ_f runs through nontrivial zeros of $L(s, f)$ and Φ is a "nice" test function. This measures the distribution of low-lying zeros *on average* over elements of large conductor.

Katz and Sarnak predict that for many families, there is a measure $W_{\mathcal{F}}$ coming from matrices such that

$$\mathrm{OLD}_{\Phi}(\mathcal{F}) = \int_{\mathbb{R}} \widehat{\Phi}(x)\widehat{W_{\mathcal{F}}}(x)dx$$

for all nice test functions Φ.

Theorem (Iwaniec–Luo–Sarnak 2000)

Assume GRH. Let Φ be a Schwarz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-2, 2)$. Let $H_k^{\pm}(N)$ denote a Hecke eigenbasis of modular newforms of weight k and root number $\epsilon_f = \pm 1$. Then

$$OLD_{\Phi}(H_k^{\pm}) = \int_{\mathbb{R}} \widehat{\Phi}(x)\widehat{W_{\mathrm{SO}(\pm)}}(x)dx$$

where $W_{\mathrm{SO}(+)} = 1 + \frac{\sin(2\pi x)}{2\pi x}$ and $W_{\mathrm{SO}(-)} = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x)$.

Connection to Murmurations

Unravelling, ILS shows that (under GRH)

$$\lim \frac{1}{\#H_k^{\pm}(N)} \sum_{f \in H_k^{\pm}(N)} \sum_{\gamma_f} \Phi\left(\frac{\gamma_f \log N}{2\pi}\right) = \int_{\mathbb{R}} \widehat{\Phi}(x) \widehat{W_{\mathrm{SO}(\pm)}}(x) dx.$$

The explicit formula relating zeros to sums over primes implies that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f \log N}{2\pi}\right) \approx \sum_{p} \frac{\lambda_f(p) \log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log N}\right).$$

Note that if $\widehat{\Phi}$ is supported on $[-\theta, \theta]$, then only primes $\leq N^{\theta}$ appear. Hence one level density behaves like

$$\mathop{\mathbb{E}}_{p \sim N^{\theta}} \mathop{\mathbb{E}}_{f \in H_k^{\pm}(N)} [\lambda_f(p) \log p / \sqrt{p}].$$

$$\mathop{\mathbb{E}}_{p \sim N^\theta} \mathop{\mathbb{E}}_{f \in H_k^\pm(N)} [\lambda_f(p) \log p / \sqrt{p}]$$

is *almost* an averaged murmuration behavior (with some prime scaling), except that murmurations concern $p \sim N$ instead of $p \sim N^\theta$. Specifically

$$\widehat{W_{\mathrm{SO}(+)}}(y) = \delta_0(y) + \frac{\mathbf{1}_{[-1,1]}(y)}{2}$$
$$\widehat{W_{\mathrm{SO}(-)}}(y) = \delta_0(y) + \frac{2 - \mathbf{1}_{[-1,1]}(y)}{2}.$$

There is a discontinuity in behavior exactly when $\pm 1 \in \mathrm{Supp}(\widehat{\Phi})$. Murmurations arise from the transition range for one-level density for known families, which is much more mysterious than $p \sim N^\theta$ for $\theta < 1$ or $\theta > 1$.

Unpublished result

In some cases, it is possible to *prove* murmuration-like phenomena from the same work that goes into one-level density results. The proofs in Iwaniec-Luo-Sarnak [ILS2000] show

$$\frac{\sum_{f\in H_k^{\pm}(1)} \Phi(N_f/X)\frac{\lambda_f(p)\sqrt{p}}{L(1,\mathrm{sym}^2 f)}}{\sum_{f\in H_k^{\pm}(1)} \Phi(N_f/X)\frac{1}{L(1,\mathrm{sym}^2 f)}} = \pm\sum_{c\geq 1}\frac{\mu^2(c)}{c^2\varphi(c)}\Phi(*),$$

which describes correlations between $\lambda_f(p)/L(1,\mathrm{sym}^2 f)$ with root numbers.

Unpublished result

Sarnak refers to this an arithmetic normalization murmuration,

$$A_\Phi^{\mathcal{F},\mathrm{arith}}(f,X) := \sum_{\pi\in\mathcal{F}} \Phi(h(\pi)/X)\frac{f(\pi)}{\mathrm{arith}(\pi)},$$

$$\mathbb{E}^{\mathrm{arith}}_{\pi\in\mathcal{F}}[f;X] := \frac{A_\Phi^{\mathcal{F},\mathrm{arith}}(f,X)}{A_\Phi^{\mathcal{F},\mathrm{arith}}(1,X)}.$$

Then the unpublished result is the arithmetically normalized murmuration

$$\mathbb{E}^{\mathrm{arith}}_{\pi\in H_k^{\pm}(1)}[\lambda_f(p)\sqrt{p};X] = \pm\sum_{c\geq 1}\frac{\mu^2(c)}{c^2\varphi(c)}\Phi(*).$$

Enter the trace formula

The key tool at the start of Iwaniec-Luo-Sarnak's arithmetically normalized murmuration is to use the Petersson trace formula

$$(*) \sum_{f \in H_k(1)} \frac{a_f(m) a_f(n)}{\|f\|^2} = \delta_{[m=n]} + 2\pi i^{-k} \sum_{c>0} \frac{S(m,n;c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Here $S(m,n;c)$ is the Kloosterman sum $\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{\frac{2\pi i}{c}(md + n\overline{d})}$ and $J_\nu(x)$ is the Bessel function of the first kind.

We don't focus on the details. Instead, we note that this formula renders averages over $H_k(1)$ as a sum over completely explicit, directly computable things.

More generally, every proved murmuration that I know uses some trace formula.

1. Zubrilina [Z2023] uses the Eichler-Selberg trace formula for fixed weight, varying level holomorphic modular forms.
2. Yesterday, Steven Wang [W2024] uses similar techniques to prove murmurations of Hecke L-functions.
3. In [BBLLD2023], we use the Eichler-Selberg trace for murmurations of fixed level, varying weight holomorphic modular forms.
4. In [LOP2023], they use Gauss sums (traces of a character on finite fields) and Poisson summation (the most classical trace formula) for murmurations of Dirichlet characters.
5. In [BLLDSHZ2024], we use a Selberg-Strömbergsson trace formula for murmurations of Maass forms of fixed level and weight with varying eigenvalue.
6. On Monday, Kimball Martin [M2024B, based on M2024A] uses a trace formula due to Yamauchi to prove (morally) a murmuration-like phenomenon of local root numbers of modular forms.

Although the details differ, the broad mechanism behind many of these proofs look the same. To radically oversimplify: start with a trace formula and then perform a sufficient amount of averaging in each non-fixed aspect.

Flaws

There are problems with relying only on trace formulas to prove murmurations.

Perhaps most obviously, we don't know of nearly enough trace formulas to explain all the murmuration phenomena that we can experimentally observe:

Like elliptic curves, or genus 2 curves, or more than half of the other phenomena that Drew mentioned in his talk on Monday.

There are also qualitative flaws. Recall the unpublished arithmetic murmuration of ILS:

$$\mathbb{E}^{\mathrm{arith}}_{\pi \in H_k^{\pm}(1)}[\lambda_f(p)\sqrt{p}; X] = \pm \sum_{c \geq 1} \frac{\mu^2(c)}{c^2 \varphi(c)} \Phi(*).$$

This is very close to the murmuration function in BBLLD2023 and BLLDSHZ2024, which is

$$\frac{\pm 1}{\zeta(2)} \sum_{\substack{a,q \in \mathbb{Z}_{>0} \\ \gcd(a,q)=1 \\ (a/q)^{-2} \in E}} \frac{\mu(q)^2}{\varphi(q)^2 \sigma(q)} \left(\frac{q}{a}\right)^3,$$

also weighted sum over point masses at something like squarefree integers and with similar weights.

But why are the functions identical in BBLLD2023 and BLLDSHZ2024, and is there an easier way to explain where the differences with ILS come from (<u>without going through the whole trace formula</u>)?

If you have a "reasonable" trace formula and perform "reasonable" averages, you can *probably* prove a murmuration.

For example, when experimenting if there were murmurations of Maass forms in the eigenvalue aspect, I looked at data coming from the Kuznetsov trace formula (here, for level 1):

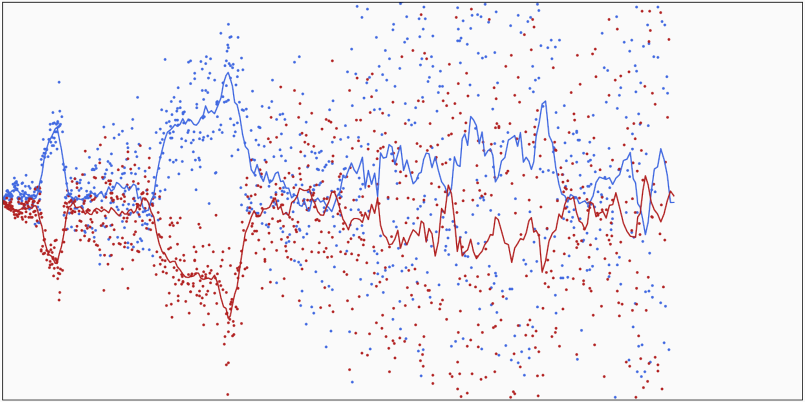
$$\sum_{j \geq 1} \frac{h(r_j)}{\cosh(\pi r_j)} \frac{\lambda_j(n)\lambda_j(m)}{\|\mu_j\|^2} + \widehat{h}(0) = \sum_{c \geq 1} \frac{S(n, m; c)}{c} k^*(4\pi\sqrt{mn}/c) + \delta_{[m=n]} k(0).$$

Here $\frac{1}{4} + r_j^2$ is the eigenvalue of the jth Maass form μ_j, h is a "nice" function, and k^* is a Bessel transform

$$k^*(x) = \frac{i}{\pi} \int_{\mathbb{R}} \big(J_{2ir}(x) - J_{-2ir}(x)\big) \frac{rh(r)}{\cosh(\pi r)} dr.$$

Choosing $n = 1$ gives a normalized sum of the mth coefficient of Maass forms, normalized by $\|\mu_j\|^2 \asymp L(1, \mathrm{Sym}^2 \mu_j)$.

Heuristic, slightly old plots of Kuznetsov trace based plots for arithmetic Maass murmurations.



In BLLDSHZ2024 we proved a version without arithmetic weights. But there appears to be an arithmetically weighted murmuration that is qualitatively similar. . . and presumably provable using the Kuznetsov trace formula (and otherwise similar analysis to BLLDSHZ2024).

(And it would be very nice if we could "just see" what the murmuration is without doing all the work, but I don't know how to do that).

Aside on applying murmurations

The Maass form data in the LMFDB is incomplete. Many forms weren't computed with sufficient precision to directly deduce the sign of the functional equation.

But murmurations are all about correlations between $a(p)$ and this sign. This semester, BBCDLLDOQV[1] have been looking into using these correlations to predict the missing signs.

Neural networks trained on Maass form data can predict the correct sign (on data of LMFDB size) with extremely high accuracy. The betting game based on murmurations is a very successful game.

[1] Joanna Bieri, Giorgi Butbaia, Edgar Costa, Alyson Deines, Kyu-Hwan Lee, David Lowry-Duda, Thomas Oliver, Yidi Qi, and Tamara Veenstra

Several explorable murmurations

Extend BBLLD2023 and BLLDSHZ2024

There are no major hurdles preventing one from extending BBLLD2023 (modular murmurations in weight aspect) to general level using either the Eichler-Selberg or Petersson trace formulas.

There *are* major hurdles in the way of directly extending BLLDSHZ2024 (Maass murmurations) to general level using the Selberg-Strömbergsson trace formula, but probably not using the arithmetical normalization in Kuznetsov.

In a different direction, it should be possible to prove murmurations for weight 1 Maass forms using Kuznetsov. (Both the weight and level generalizations using non-arithmetic normalization should be attainable with a forthcoming trace formula of BBKLLDSH202?).

$\mathrm{GL}(3)$ murmuration — symmetric square

Here is a strategy to prove a $\mathrm{GL}(3)$-type murmuration: if f is a holomorphic modular form with coefficients $a_f(n)$, then $\mathrm{Sym}^2(f)$ is a $\mathrm{GL}(3)$ form, and $a_{\mathrm{Sym^2f}}(p) = a_f(p^2)$.

Thus applying a $\mathrm{GL}(2)$ type trace formula (such as Eichler-Selberg or Petersson) to study murmuration behavior across $a_f(p^2)$ also identifies murmuration behavior across $a_{\mathrm{Sym}^2 f}(p)$.

(BBLLDSH is currently investigating this).

The analogous construction using the Kuznetsov trace formula to study murmurations in symmetric square lists of Maass forms is also (probably) provable.

More general sparse subsequences (maybe)

More generally, one could consider arbitrary sparse subsequences. Steven Creech is studying correlations of $a_f(q(n))$ with root numbers, where $q(n)$ is an irreducible quadratic polynomial.

One could try to study $a_f(p^m)$ to understand $a_{\mathrm{Sym^m f}}(p)$ and thus study higher $\mathrm{GL}(n)$ behavior. (I think this is hard, but I'm not completely sure).

Murmurations where RH is false

It should also be possible to prove murmurations for a family of L-series that don't satisfy the Riemann Hypothesis.

To a weight $k + \frac{1}{2}$ cuspidal modular form f, we can associate a Dirichlet series $D(s, f)$ that has a functional equation $N^s D(s, f) G(s) = \varepsilon N^{1-s} D(1 - s, \widetilde{f}) G(1 - s)$, analytic continuation, and so on; but they don't have RH.

Nonetheless, a version of the Petersson trace formula applies and the analysis seems similar to ILS2000 and BBLLD2023. I expect provable murmuration phenomena here too, showing that murmurations are independent of RH.

(BBLLDSH is currently investigating this as of this week).

Similarly, a version of Kuznetsov applies to half-integral weight Maass forms, and we should expect murmuration behavior there to be provable.

Thank you very much.

Please note that these slides are (or will soon be) available on my website (davidlowryduda.com).

[BBLLD2023]: Bober, Booker, Lee, Lowry-Duda. *Murmurations of modular forms in the weight aspect*. arXiv:2310.07746

[BLLDSHZ2024]: Booker, Lee, Lowry-Duda, Seymour-Howell, Zubrilina. *Más Maass Murmurations* (title in development). arXiv:2409.00765

[ILS2000]: Iwaniec, Luo, Sarnak. *Low lying zeros of families of L-functions*. IHES. 2000.

[LOP2023]: Lee, Kyu-Hwan, Thomas Oliver, and Alexey Pozdnyakov. *Murmurations of Dirichlet characters*. arXiv:2307.00256.

[M2024A]: Martin, Kimball. *Distribution of local signs of modular forms and murmurations of Fourier coefficients*. arXiv: 2409.02338

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[M2024B]: Martin, Kimball. *Slides during Murmurations workshop at SCGP*. 2024.

[W2024]: Wang, Stephen. *Slides during Murmurations workshop at SCGP*. 2024.

[Z2023]: Zubrilina, Nina. *Murmurations*. arXiv:2310.07681