

# HECKE ALGEBRAS

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9 SEPTEMBER 2024

I want to give complete details about a set of descriptions in Bump [Bum98] concerning Hecke operators and Hecke algebras. I like that Bump's description is so light on annoying computation. But the problem is that I think too much is swept under the rug.

So here, I fill in more details (but mostly keep the same presentation as there).

## 1. DOUBLE COSETS

If  $H$  is a group acting on the left on a set  $X$ , the set of orbits of  $X$  under this action is denoted by  $H \backslash X$ . If  $H$  acts on  $X$  on the right, we denote the set of orbits by  $X/H$ . When  $H_1$  acts on  $X$  on the left and  $H_2$  acts on  $X$  on the right, *and* if the actions are compatible in the sense that

$$(h_1x)h_2 = h_1(xh_2)$$

for all  $h_1 \in H_1, x \in X$ , and  $h_2 \in H_2$ , then we again have a nicely defined dual-action and we have orbits under this dual action. Two elements  $x$  and  $y$  in  $X$  are in the same orbit if and only if  $x = h_1yh_2$  for some  $h_1$  and  $h_2$ . We denote these orbits by  $H_1 \backslash X/H_2$ .

One easy source of compatible actions are when  $H_1$  and  $H_2$  are subgroups of a group  $G = X$ . Then  $H_1 \backslash G/H_2$  is the set of double cosets  $H_1gH_2$ .

Below, we let  $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$ .

**Proposition 1.** *Take  $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ . The double coset  $\Gamma(1)\alpha\Gamma(1)$  is a finite union of right cosets*

$$\Gamma(1)\alpha\Gamma(1) = \bigcup_i \Gamma(1)\alpha_i, \quad \alpha_i \in \mathrm{GL}(2, \mathbb{Q})^+.$$

*The number of right cosets is equal to  $[\Gamma(1) : \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)] < \infty$ .*

For this proof, we follow Bump closely and add only a couple of additional details.

*Proof.* Multiplying on the right by  $\alpha^{-1}$  is a bijection of  $\mathrm{GL}(2, \mathbb{Q})^+$  onto itself. This gives an obvious bijection between the sets of orbits

$$\Gamma(1) \backslash \Gamma(1)\alpha\Gamma(1) \longleftrightarrow \Gamma(1) \backslash \Gamma(1)\alpha\Gamma(1)\alpha^{-1}.$$

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This work was supported by the Simons Collaboration in Arithmetic Geometry, Number Theory, and Computation via the Simons Foundation grant 546235.

Any element  $U$  of  $\Gamma(1)\backslash\Gamma(1)\alpha\Gamma(1)\alpha^{-1}$  can be written as  $U = \Gamma(1)\gamma\alpha u\alpha^{-1} = \Gamma(1)\alpha u\alpha^{-1} \in \alpha\Gamma(1)\alpha^{-1}$  for some  $\gamma, u \in \Gamma(1)$ . Observe that two orbits  $U = \Gamma(1)\alpha u\alpha^{-1}$  and  $V = \Gamma(1)\alpha v\alpha^{-1}$  are the same when

$$\begin{aligned} \Gamma(1)\alpha u\alpha^{-1} = \Gamma(1)\alpha v\alpha^{-1} &\iff \Gamma(1)\alpha u = \Gamma(1)\alpha v \\ &\iff \Gamma(1)\alpha u v^{-1}\alpha^{-1} = \Gamma(1) \\ &\iff \alpha u v^{-1}\alpha^{-1} \in \Gamma(1) \cap \alpha\Gamma(1)\alpha^{-1}. \end{aligned}$$

Thus we have directly seen that

$$\Gamma(1)\backslash\Gamma(1)\alpha\Gamma(1)\alpha^{-1} \cong (\Gamma(1) \cap \alpha\Gamma(1)\alpha^{-1})\backslash\alpha\Gamma(1)\alpha^{-1}.$$

Conjugating by  $\alpha$  gives a bijection between this set of cosets and the set of cosets

$$(\alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1))\backslash\Gamma(1).$$

This shows that the cardinality is  $[\Gamma(1) : \alpha^{-1}\Gamma(1)\alpha \cap \Gamma(1)]$ . It only remains to show that this is finite. This follows from Lemma 1.4.1 of Bump, which we give below.  $\square$

**Lemma 2** (Lemma 1.4.1 of Bump). *Let  $\Gamma$  be a congruence subgroup of  $\mathrm{SL}(2, \mathbb{Z})$ . Let  $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ . Then  $\alpha^{-1}\Gamma\alpha \cap \Gamma(1)$  is a congruence subgroup.*

(This is an easy proof).

Now let  $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$  be any congruence subgroup.

**Corollary 3.** *Take  $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ . The double coset  $\Gamma\alpha\Gamma$  is a finite union of right cosets*

$$\Gamma\alpha\Gamma = \bigcup \Gamma\alpha_i, \quad \alpha_i \in \mathrm{GL}(2, \mathbb{Q})^+.$$

*The number of right cosets is equal to  $[\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma]$ .*

*Proof.* Structurally, the proof is identical until we want to show that  $[\Gamma : \alpha^{-1}\Gamma\alpha \cap \Gamma] < \infty$ . This follows from a general result: if  $[G : H_1] < \infty$  and  $[G : H_2] < \infty$ , then  $[G : H_1 \cap H_2] < \infty$ . We prove this in Lemma 4 below.

We apply this repeatedly to a lattice of subgroups formed from  $\Gamma(1)$  and  $\alpha^{-1}\Gamma(1)\alpha$ . Note that  $[\Gamma(1) : \Gamma] < \infty$  and  $[\alpha^{-1}\Gamma(1)\alpha : \alpha^{-1}\Gamma\alpha] < \infty$  because  $\Gamma$  is a congruence subgroup. And as shown above,  $\Gamma(1) \cap \alpha^{-1}\Gamma(1)\alpha$  has finite index in both  $\Gamma(1)$  and  $\alpha^{-1}\Gamma(1)\alpha$ .

Applying Lemma 4 to  $\Gamma$  and  $\Gamma(1) \cap \alpha^{-1}\Gamma(1)\alpha$  in  $\Gamma(1)$  shows that  $\Gamma \cap \alpha^{-1}\Gamma(1)\alpha$  has finite index in  $\Gamma(1)$ . Applying Lemma 4 to  $\Gamma(1) \cap \alpha^{-1}\Gamma(1)\alpha$  and  $\alpha^{-1}\Gamma\alpha$  inside  $\alpha^{-1}\Gamma(1)\alpha$  shows that  $\Gamma(1) \cap \alpha^{-1}\Gamma\alpha$  has finite index in  $\alpha^{-1}\Gamma(1)\alpha$ .

Finally, applying the lemma once more to  $\Gamma(1) \cap \alpha^{-1}\Gamma\alpha$  and  $\Gamma \cap \alpha^{-1}\Gamma(1)\alpha$  (which we've now shown have finite index in either  $\Gamma(1)$  or  $\alpha^{-1}\Gamma(1)\alpha$ ) shows that  $\Gamma \cap \alpha^{-1}\Gamma\alpha$  has finite index in  $\Gamma(1)$  and  $\alpha^{-1}\Gamma(1)\alpha$ . (And thus also in  $\Gamma$ ).  $\square$

This proof is annoying. It's slightly clearer if one draws the lattice of subgroups. Alternately, here is the idea: we start with  $\Gamma(1)$  and  $\alpha^{-1}\Gamma(1)\alpha$ . We twice pass to  $\Gamma$  (and use finite index because it's a congruence subgroup) to get results on smaller groups. We apply the previous result once to understand

$\Gamma(1) \cap \alpha^{-1}\Gamma(1)\alpha$ . We then repeatedly (three times — once for the first group, once for the second group, and then once together) intersect things with  $\Gamma$  and things with  $\Gamma(1)$  to pass to the smaller subgroups.

**Lemma 4.** *If  $[G : H_1] < \infty$  and  $[G : H_2] < \infty$ , then  $[G : H_1 \cap H_2] < \infty$ .*

*Proof.* The group  $G$  acts on the cross-product of orbits  $G/H_1 \times G/H_2$  by multiplication in each coordinate:  $g(aH_1, bH_2) = (gaH_1, gbH_2)$ . The stabilizer of the orbit of  $(1, 1)$ , i.e. of  $(H_1, H_2)$ , consists of those  $g \in G$  such that  $gH_1 = H_1$  and  $gH_2 = H_2$ . Of course,  $gH_1 = H_1 \iff g \in H_1$ , similarly for  $H_2$ . Thus the stabilizer is  $H_1 \cap H_2$ .

By the orbit-stabilizer theorem,  $[G : H_1 \cap H_2] = |\mathcal{O}(H_1, H_2)|$ , the size of the orbit of  $(H_1, H_2)$  under multiplication by  $G$ . As both  $H_1$  and  $H_2$  have finite index in  $G$ , the size of this orbit is at most the product of the indices. Thus  $[G : H_1 \cap H_2] \leq [G : H_1][G : H_2]$ .  $\square$

## 2. HECKE ALGEBRA

We define the Hecke algebra  $\mathcal{R}$  associated to a congruence subgroup  $\Gamma$ . As an abelian group,  $\mathcal{R}$  is the free abelian group on symbols  $T_\alpha = [\Gamma\alpha\Gamma]$  as  $\alpha$  runs through a complete set of representatives for  $\Gamma \backslash \mathrm{GL}(2, \mathbb{Q})^+ / \Gamma$ .

Defining the multiplication on  $\mathcal{R}$  is annoying. It's nonobvious how to define multiplication on  $\mathcal{R}$  directly — a lot of the complexity of various introductions to Hecke algebras comes what path is taken to define multiplication. The approach we take here is to define an action of  $\mathcal{R}$  on a particular set, and the necessary structure for this to be a well-defined action will give our multiplicative structure.

**2.1. Defining an action.** Given any group  $G$ , a (right)  $G$ -module  $M$  is an abelian group with a  $\mathbb{Z}$ -linear  $G$  action. I emphasize that we will use right multiplication here, which (in my experience) is much less common.

For a  $G$ -module  $M$  and a subgroup  $\Gamma \leq G$ , write  $M^\Gamma$  to be the fixed points of  $M$  under the action of  $G$ ,

$$M^\Gamma := \{m \in M : m\gamma = m \forall \gamma \in \Gamma\}.$$

Suppose that for any  $g \in G$ , we have  $[\Gamma : \Gamma \cap g^{-1}\Gamma g] < \infty$ . The argument from Proposition 1 and its corollary shows that

$$\Gamma g \Gamma = \bigcup \Gamma g_i, \quad g_i \in G. \quad (1)$$

We now define the action of the element  $[\Gamma g \Gamma]$  (thought of as an element of  $\mathcal{R}(G, \Gamma)$ , the abelian free group on the symbols  $[\Gamma\alpha\Gamma]$  where  $\alpha$  ranges over elements of  $\Gamma \backslash G / \Gamma$ ) on an element  $m \in M^\Gamma$  as

$$m | [\Gamma g \Gamma] := \sum_i m g_i,$$

where the  $g_i$  are as in (1).

**Proposition 5.** *We prove two properties of this action. Let  $m \in M^\Gamma$  and  $g \in G$  as above.*

- (1)  $m \mid [\Gamma g \Gamma]$  depends only on the double coset  $\Gamma g \Gamma$ .  
(2)  $m \mid [\Gamma g \Gamma] \in M^\Gamma$ .

To make sure this makes sense: the item  $[\Gamma g \Gamma]$  is a formal symbol, a basis element in an abelian free group on symbols. The content of this proposition is that using any symbol from the same double coset is equivalent, and the action yields an element of  $M$  stabilized by  $\Gamma$ .

*Proof.* Suppose  $\Gamma g \Gamma = \bigcup \Gamma g'_i$  for  $g'_i = \gamma_i g_i$ . Any such choice is an equally valid way of writing  $\Gamma g \Gamma$  as a union of right cosets. Then

$$\sum_i m g'_i = \sum_i (m \gamma_i) g_i = \sum_i m g_i,$$

where we have used that  $m \in M^\Gamma$ . This proves the first claim.

For the second claim, the point is that  $\bigcup \Gamma g_i$  is a complete set of coset representatives for  $\Gamma g \Gamma$ . Multiplying on the right by  $h \in G$  doesn't change  $\Gamma g \Gamma$ , and hence  $\{\Gamma g_i h\}$  is a permutation  $\{\Gamma g_j\}$  of the coset representatives. This means that

$$(m \mid [\Gamma g \Gamma])h = \left( \sum_i m g_i \right) h = \sum_i m g_i h = \sum_j m g_j = m \mid [\Gamma g \Gamma],$$

proving the second claim.  $\square$

Now let  $\mathbb{Z}[\Gamma \backslash G]$  be the free abelian group on cosets  $[\Gamma g]$ . The group  $G$  acts on the right on  $\mathbb{Z}[\Gamma \backslash G]$  by right multiplication. Consider the map from  $\mathcal{R}(G, \Gamma)$ , the free abelian group on double cosets  $[\Gamma g \Gamma]$  with  $g$  ranging across representatives  $\Gamma \backslash G / \Gamma$ , to  $\mathbb{Z}[\Gamma \backslash G]^\Gamma$  given by

$$\begin{aligned} F: \mathcal{R}(G, \Gamma) &\longrightarrow \mathbb{Z}[\Gamma \backslash G]^\Gamma \\ [\Gamma g \Gamma] &\mapsto \sum [\Gamma g_i], \end{aligned} \tag{2}$$

where  $\Gamma g \Gamma = \bigcup \Gamma g_i$ . The double coset  $[\Gamma g \Gamma]$  is a (right) orbit of the single coset  $\Gamma \backslash G$  under  $\Gamma$ , which are exactly those elements of  $\mathbb{Z}[\Gamma \backslash G]$  invariant under right multiplication by  $\Gamma$ . Thus  $F$  is an isomorphism.

**Proposition 6.** *The free abelian group  $\mathcal{R}(G, \Gamma)$  on double cosets  $[\Gamma g \Gamma]$  with  $g$  ranging across representatives  $\Gamma \backslash G / \Gamma$  is isomorphic to the submodule of  $\mathbb{Z}[\Gamma \backslash G]$  fixed under right multiplication by  $\Gamma$ . (The isomorphism is as groups).*

**Theorem 7.** *There is a product on  $\mathcal{R}(G, \Gamma)$  making it into an associative ring, such that for any (right)  $G$ -module  $M$  we have  $M^\Gamma$  is a right  $\mathcal{R}(G, \Gamma)$  module.*

*Proof.* Take  $M = \mathbb{Z}[\Gamma \backslash G]$  initially. Write  $\Gamma g \Gamma = \bigcup \Gamma g_i$  and  $\Gamma h \Gamma = \bigcup \Gamma h_j$ .

We claim that

$$\sum_{i,j} [\Gamma g_i h_j] \in M^\Gamma.$$

To see this, note that  $F([\Gamma g \Gamma]) = \sum_i [\Gamma g_i] \in M^\Gamma$ . Now compute

$$\sum_i [\Gamma g_i] \mid [\Gamma h \Gamma] = \sum_{i,j} [\Gamma g_i h_j]$$

by the definition of the right action of  $\mathcal{R}(G, \Gamma)$ . By Proposition 5, this is in  $M^\Gamma$  and is well-defined.

The point that we've been working towards this whole time is that we can define

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = F^{-1} \left( \sum_{i,j} [\Gamma g_i h_j] \right),$$

where  $F$  is the isomorphism from (2). This is clearly associative as associativity in  $G$  is associative, and the isomorphism preserves this. The multiplicative unit is  $[\Gamma 1 \Gamma]$ .

And more generally, if  $M$  is any right  $G$ -module, then for  $m \in M^\Gamma$  we have

$$m \mid [\Gamma g \Gamma] \mid [\Gamma h \Gamma] = \sum m g_i \mid [\Gamma h \Gamma] = \sum m g_i h_j = m \mid ([\Gamma g \Gamma] \cdot [\Gamma h \Gamma]).$$

□

As used here, the actual multiplication is defined through  $F^{-1}$ . This is annoying.

We now give an explicit formula. To do so, we suppose we have a fixed set of representatives  $\sigma$  ranging over  $\Gamma \backslash G / \Gamma$ .

**Proposition 8.** *Write  $\Gamma g \Gamma = \cup \Gamma g_i$  and  $\Gamma h \Gamma = \cup \Gamma h_j$ . Then*

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \sum_{\sigma \in \Gamma \backslash G / \Gamma} m(h, g; \sigma) [\Gamma \sigma \Gamma],$$

where  $m(h, g; \sigma)$  is the number of pairs  $(i, j)$  such that  $\Gamma g_i h_j = \Gamma \sigma$ .

*Proof.* We define  $[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = F^{-1}(\sum_{i,j} [\Gamma g_i h_j])$ , where  $F$  takes  $[\Gamma g \Gamma]$  to  $\sum [\Gamma g_i]$  for all  $i$  in an index set  $I$  (similarly for  $[\Gamma h \Gamma]$  and  $[\Gamma h_j]$  for  $j \in J$ ) and where the sum is over all  $(i, j) \in I \times J$ .

This is a matter of identifying the preimage correctly. Any element in  $\mathcal{R}(G, \Gamma)$  has the form  $\sum m_\sigma [\Gamma \sigma \Gamma]$  for some integers  $m_\sigma$ . How do we determine what  $m_\sigma$  is?

For each  $\sigma$  in the preimage (i.e. where  $m_\sigma > 0$ ), it must be that  $\Gamma \sigma \Gamma = \cup_{i,j \in A} \Gamma g_i h_j$ , for some indices  $i, j$  in some index set  $A$ . It's not necessary for  $A = I \times J$  — it's only necessary that the union is stable under right multiplication by  $\Gamma$ , which could be in a smaller index set.

Thinking of  $\Gamma \sigma \Gamma = (\Gamma \sigma) \Gamma$  as a right orbit of  $\Gamma \sigma$ , it suffices to count how often  $\Gamma \sigma = \Gamma g_i h_j$  for any pair  $(i, j)$ . For each pair  $(i, j)$  with  $\Gamma \sigma = \Gamma g_i h_j$ , the preimage of  $\sum_{i,j} [\Gamma g_i h_j]$  must include at least 1  $[\Gamma \sigma \Gamma]$ . As  $\sigma$  ranges across representatives of  $(\Gamma \backslash G) / \Gamma$  (where we've added parentheses for emphasis), no other elements require any further  $[\Gamma \sigma \Gamma]$ . Thus we find that  $m_\sigma = m(h, g; \sigma)$  is the number of pairs  $(i, j)$  such that  $\Gamma g_i h_j = \Gamma \sigma$ .

Stated differently: for any  $\sigma$  with  $\Gamma \sigma = \Gamma g_i h_j$  for some  $(i, j)$  pair, the full orbit  $\Gamma \sigma \Gamma$  will be given by a union  $\cup \Gamma g_i h_j$  across some index set. Instead of counting each element of the index set, we only count the "trivial" elements of the orbits under  $\Gamma$ , given by  $\Gamma \sigma \cdot 1$ . The other elements are handled by the averaging of  $F$ , and the fact that  $\mathcal{R}(G, \Gamma)$  is isomorphic to  $\mathbb{Z}[\Gamma \backslash G]^\Gamma$  means that we don't have to worry about the preimage not existing. □

**Corollary 9.** Write  $\Gamma g\Gamma = \bigcup \Gamma g_i$ ,  $\Gamma h\Gamma = \bigcup \Gamma h_j$ , and  $\Gamma f\Gamma = \bigcup \Gamma f_k$ . Then

$$[\Gamma g\Gamma] \cdot [\Gamma h\Gamma] \cdot [\Gamma f\Gamma] = \sum_{\sigma \in \Gamma \backslash G/\Gamma} m(h, g, f; \sigma) [\Gamma \sigma \Gamma]$$

where  $m(h, g, f; \sigma)$  is the number of triples  $(i, j, k)$  such that  $\Gamma g_i h_j f_k = \Gamma \sigma$ .

*Proof.* As above, we define  $[\Gamma g\Gamma] \cdot [\Gamma h\Gamma] \cdot [\Gamma f\Gamma] = F^{-1}(\sum_{i,j,k} [\Gamma g_i h_j f_k])$ . The idea is the same: it's sufficient to account for the trivial elements of orbits,  $\Gamma \sigma \cdot 1$ . And these give one  $[\Gamma \sigma \Gamma]$  for each  $\sigma$  with  $\Gamma \sigma = \Gamma g_i h_j f_k$ , exactly as before.  $\square$

The statement of Corollary 9 is stated without proof in Bump as a way to note that multiplication in a Hecke algebra is associative.

**Remark 10.** Bump defines only the Hecke algebra  $\mathcal{R}(\mathrm{GL}(2, \mathbb{Q})^+, \mathrm{SL}(2, \mathbb{Z}))$ , while the presentation here is more general. I don't know much about Hecke algebras apart from their use in automorphic forms and representations.

### 3. ACTING ON MODULAR FORMS

To end, we bring this discussion back to the context of modular forms, as in the first chapter of Bump [Bum98].

Let  $\mathcal{R}$  denote the Hecke algebra  $\mathcal{R}(\mathrm{GL}(2, \mathbb{Q})^+, \mathrm{SL}(2, \mathbb{Z}))$ . Let  $M_k = M_k(\mathrm{SL}(2, \mathbb{Z}))$  denote the space of holomorphic modular forms of level 1 and weight  $k$ . Then  $M_k$  is a (right)  $\mathrm{GL}(2, \mathbb{Q})^+$  module under the slash operator

$$(f | \gamma)(z) := (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z), \quad \gamma \in \mathrm{GL}(2, \mathbb{Q})^+.$$

Verifying that  $(f | \gamma) | \gamma' = f | (\gamma \gamma')$  is a straightforward direct calculation.

By Theorem 7,  $M_k^{\mathrm{SL}(2, \mathbb{Z})}$  (i.e. the space of modular forms in  $M_k$  fixed by the action of  $\mathrm{SL}(2, \mathbb{Z})$ ) is a right  $\mathcal{R}$  module. The definition of modularity gives immediately that  $M_k^{\mathrm{SL}(2, \mathbb{Z})} = M_k$ , so this shows that  $M_k$  is a right  $\mathcal{R}$  module. For  $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ , we denote the element  $\Gamma \alpha \Gamma = \mathrm{SL}(2, \mathbb{Z}) \alpha \mathrm{SL}(2, \mathbb{Z}) \in \mathcal{R}$  by  $T_\alpha$ . For  $f \in M_k$  and  $\alpha \in \mathrm{GL}(2, \mathbb{Q})^+$ , the right action is given by

$$f | T_\alpha := f | [\mathrm{SL}(2, \mathbb{Z}) \alpha \mathrm{SL}(2, \mathbb{Z})] = \sum_i f | \alpha_i,$$

where

$$\mathrm{SL}(2, \mathbb{Z}) \alpha \mathrm{SL}(2, \mathbb{Z}) = \bigcup_i \alpha_i$$

as a disjoint union (cf. Proposition 1). Various properties claimed or shown in Bump follow from our discussion:

- (1) Proposition 5 shows that  $f | T_\alpha$  depends only on the equivalence class of  $\alpha$  (and not on the choice of representatives  $\alpha_i$ ), and further that  $f | T_\alpha \in M_k$ .
- (2) By Proposition 8, we have that

$$(f | T_\alpha) | T_\beta = \sum_{\sigma \in \mathrm{SL}(2, \mathbb{Z}) \backslash \mathrm{GL}(2, \mathbb{Q})^+ / \mathrm{SL}(2, \mathbb{Z})} m(\alpha, \beta; \sigma) (f | T_\sigma).$$

This is a proof of equation 4.4 in Bump [Bum98] (although in Bump this falls out as a consequence of the definition).

- (3) The action of  $\mathcal{R}$  on  $M_k$  is associative! The associativity (and other properties) are baked into the formulation given in this note: multiplication in  $\mathcal{R}(G, \Gamma)$  comes from the right action of  $\mathcal{R}(G, \Gamma)$  on  $\mathbb{Z}[\Gamma \backslash G]^\Gamma$ , and this right action is trivially associative because group multiplication is associative. As noted above, Corollary 9 proves an unproven assertion in Bump.

I note that neither the presentation in Bump or the presentation here makes it obvious that the Hecke algebra  $\mathcal{R}$ , or those associated to congruence subgroups, is commutative. This structural result is remarkable, but I don't discuss that further here.

#### REFERENCES

- [Bum98] Daniel Bump. *Automorphic forms and representations*, volume 55. Cambridge University Press, 1998.