COMPUTING PETERSSON INNER PRODUCT

DLD

Suppose that $f(z) = \sum_{n \geq 1} A(n) n^{\frac{k-1}{2}} e(nz)$ is a weight k, nebentypus χ Hecke eigen newform on $\Gamma_0(N)$, where χ is primitive mod $N_0 \mid N$. I assume that the coefficients $A(n)$ are the Hecke eigenvalues.

The classic Rankin–Selberg construction shows that

$$\int_{\Gamma_0(N)\backslash\mathcal{H}} |y^{\frac{k}{2}} f(z)|^2 E(z,s) \frac{dx\,dy}{y^2} = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n \geq 1} \frac{|A(n)|^2}{n^s}, \tag{1}$$

where $E(z,s)$ is the $\Gamma_0(N)$ Eisenstein series

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \mathrm{Im}(\gamma z)^s.$$

I'm happy to talk more about this, or one can look at §1.6 of Bump's *Automorphic Forms and Representations* for a small overview of how this works.

The Eisenstein series series has a residue at $s = 1$, with residue equal to

$$\frac{1}{V} := \frac{1}{\mathrm{vol}(\Gamma_0(N)\backslash\mathcal{H})}.$$

Taking residues in (1) thus shows that

$$\langle f, f \rangle = \frac{1}{V} \int_{\Gamma_0(N)\backslash\mathcal{H}} |y^{\frac{k}{2}} f(z)|^2 \frac{dx\,dy}{y^2} = \frac{\Gamma(k)}{(4\pi)^k} \operatorname*{Res}_{s=1} \sum_{n \geq 1} \frac{|A(n)|^2}{n^s}. \tag{2}$$

I note that one can choose to normalize the Petersson inner product by the volume (as I have) or not — but this makes my choice of normalization clear.

This reduces computing the Petersson inner product to the evaluation of the residue of the Dirichlet series in the RHS of (2). To evaluate this, we apply a few identities (which can be found on pages 137 and 138 of Iwaniec–Kowalski) We define

$$L(s, \mathrm{Ad}^2 f) = \frac{L(s, f \otimes \overline{f})}{\zeta(s)},$$

where

$$L(s, f \otimes \overline{f}) = \zeta^{(N)}(2s) \sum_{n \geq 1} \frac{|A(n)|^2}{n^s}$$

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is essentially Dirichlet series appearing in (2). (This combines equation 5.98 in IK with an unlabelled equation on pg 133 of IK). Together, these show that

$$\begin{aligned}\langle f, f\rangle &= \frac{\Gamma(k)}{(4\pi)^k} \operatorname*{Res}_{s=1} \frac{\zeta(s)L(s, \mathrm{Ad}^2 f)}{\zeta^{(N)}(2s)} \\ &= \frac{\Gamma(k)}{(4\pi)^k} \operatorname*{Res}_{s=1} \frac{\zeta^{(N)}(s)L^{(N)}(s, \mathrm{Ad}^2 f)}{\zeta^{(N)}(2s)} \left(\prod_{p \mid N} \sum_{j \geq 0} \frac{|A(p^j)|^2}{p^{js}} \right) \\ &= \frac{\Gamma(k)}{(4\pi)^k} \frac{L^{(N)}(1, \mathrm{Ad}^2 f)}{\zeta^{(N)}(2)} \left(\prod_{p \mid N} \sum_{j \geq 0} \frac{|A(p^j)|^2}{p^j} \right) \operatorname*{Res}_{s=1} \zeta^{(N)}(s).\end{aligned}$$

One can quickly verify that

$$\operatorname*{Res}_{s=1} \zeta^{(N)}(s) = \prod_{p \mid N} (1 - p^{-1}).$$

In total, we have computed that

$$\langle f, f\rangle = \left(\frac{\Gamma(k)}{(4\pi)^k} \frac{\prod_{p \mid N}(1 - p^{-1})}{\zeta^{(N)}(2)} \right) L^{(N)}(1, \mathrm{Ad}^2 f) \left(\prod_{p \mid N} \sum_{j \geq 0} \frac{|A(p^j)|^2}{p^j} \right).$$

In this expression, the first parenthesesized term is easily computable. The middle is the computation of only the good factors of a convergent L-function, which (I believe) can be done quickly using the Euler product — and in particular using that we only require the Euler factors at primes away from the level N. And the last factor is relatively sparse and rapidly approximated — but is ultimately a numerical approximation.

In practice, the last factor is actually expressible as a finite Euler product consisting of the behavior at bad primes, but actually determining that behavior is subtle and annoying. In most cases, I think it's numerically nicer to directly estimate it.

I also note that it's possible to rigorously bound the error from the last term using the rapid convergence and the Deligne bound for the coefficients.

References