Murmurations and Maass form L-functions

Members of ICERM discussion group on Maass forms

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1 Introduction

The goal of this work to discover whether there is a murmuration phenomenon for the L-functions associated to Maass forms. These are non-holomorphic modular forms which are eigenforms for a suitable Hecke algebra. The associated L-function has analytic continuation, Euler product and functional equation, but, in general, is not expected to arise from a Galois representation or from a motive. We shall consider the simplest case of level 1 and weight 0, that is forms for $\mathrm{SL}_2(Z)$.

2 The Kuznetsov trace formula

This is an analogue for Maass forms of Petersson's formula for holomorphic cusp forms. It involves the highly oscillatory Kloosterman sums. The formula itself is complicated to state. We give the version of Zagier (which is reproduced in section 1, Theorem 1.9 of Joyner's article). The version that has been used extensively for arithmetic applications is in the paper of Deshouillers and Iwaniec.

The Kloosterman sum is defined by

$$S(m,n;c) \ = \ \sum_{d \bmod c}{}^{*} e(\frac{md+n\overline{d}}{c})$$

where the sum ranges over residue classes d prime to c, and $\overline{d}$ satisfies $d\overline{d} \equiv 1 \bmod c$. Also, we use the usual notation

$$e(z) \ = \ \exp\{2\pi i z\}.$$

The eigenvalues of the Laplacian are written $\lambda_j = \frac{1}{4} + ir_j^2$.

Let h be an even, holomorphic function in the horizontal strip $|Im(z)| \le 1/2+\epsilon$. Define the transform

$$k^*(x) \ = \ \frac{i}{\pi}\int_{-\infty}^{\infty}(J_{2ir}(x) - J_{-2ir}(x))\frac{rh(r)}{cosh(\pi r)}dr.$$

Then

$$\sum_{j=1}^{\infty} \frac{h(r_j)}{cosh(\pi r_j)} \rho_j(n)\overline{\rho_j(m)} \ + \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(n/m)^{it}\sigma_{-2it}(n)\sigma_{2it}(m)}{|\zeta(1+2it)|^2} h(t)dt \qquad (1)$$

$= \sum_{c=1}^{\infty} \frac{1}{c} S(n,m;c) k^*(4\pi \frac{\sqrt{mn}}{c}) \ + \ \frac{4}{\pi}\delta_{mn} k(0).$

To use this formula, we could choose $m = 1$ so that we would get a normalized sum of Maass form coefficients (weighted by the first coefficient).

References

[1] J.-M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, *Invent. Math.*, **70**(1982), 219-288.

[2] D. Joyner, On the Kuznetsov-Bruggeman formula for a Hilbert modular surface having one cusp. Math Z. **203**(1990), 59–104.