Lattice points and sums of Fourier coefficients of modular forms

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Gauss circle problem

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Smoothed estimates and the Gauss sphere problem

Lattice points on hyperboloids

Much of this talk can be considered as a variety of approaches to problems of the flavor of the Gauss circle problem.

Gauss circle problem

How many integer lattice points are contained in a circle of radius $\sqrt{N}$ centered at the origin? Equivalently, how many integer solutions are there to $X^2 + Y^2 \leq N$?

I'll use $S_2(N)$ to denote this number.

We can phrase this in terms of

$$r_2(n) := \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\},$$

as then

$$S_2(N) = \sum_{n \leq N} r_2(n).$$

This is a very old problem, and a very famous problem. Over 200 years ago, Gauss noted that $S_2(N)$ is approximately the volume of the ball, and further

$$S_2(N) - \operatorname{Vol} B(\sqrt{N}) \ll \sqrt{N}.$$

Conjecture

$$S_2(N) - \operatorname{Vol} B(\sqrt{N}) \ll N^{\frac{1}{4}+\epsilon}.$$

Most approaches to this problem have used a combination of harmonic analysis, modular forms, and (variants of) the circle method. The best current bound is due to Heath-Brown [HB99], who showed that

$$S_2(N) - \operatorname{Vol} B(\sqrt{N}) \ll N^{\frac{131}{416}+\epsilon}.$$

I'm going to describe a few approaches to this problem, and its variants, in this talk.

Around the start of the 20th century, Sierpiński used basic harmonic analysis to show the first improvement over Gauss, that

$$S_2(N) - \operatorname{Vol} B(\sqrt{N}) \ll N^{\frac{1}{3}}.$$

The argument is very direct.

Let $\chi_t(x)$ denote the characteristic function of the disk of radius t centered at the origin. Fix $\delta < 1$ (which we'll choose in a moment), define $p : \mathbb{R}^2 \longrightarrow [0, \infty)$ by $\frac{1}{\pi\delta^2}\chi_\delta(x)$, and define the smoothed function

$$f_t(x) = \chi_t * p(x) = \int_{\mathbb{R}^2} \chi_t(x-y)p(y)dy.$$

Then $f_t(x)$ is 1 if $\|x\| \leq t - \delta$, is 0 if $\|x\| \geq t + \delta$, and is between 0 and 1 between.

The function f_t is a smoothed indicator function for points in the disk. Define

$$S(t) := \sum_{z \in \mathbb{Z}^2} f_t(z).$$

Then

$$S(\sqrt{n} - \delta) \leq S_2(n) \leq S(\sqrt{n} + \delta),$$

and so understanding the smoothed sum S gives approximations for our desired sum S_2.

The smoothed sum S is sufficiently smooth to allow Poisson summation, so in fact what one does is compute

$$S(t) = \sum_{z \in \mathbb{Z}^2} \widehat{f}_t(z) = \sum_{z \in \mathbb{Z}^2} \widehat{\chi}_t(z) \widehat{p}(z).$$

The Fourier transforms can be explicitly computed, and are given (essentially) by the J_1 Bessel function.

The main term comes from $\widehat{\chi}_t(0) = \pi t^2$ and $\widehat{p}(0) = 1$, and the error term comes from trying to optimize the choice ot δ to minimize the error from the rest of the summation.

Mean square estimates

Hardy and Littlewood showed that *on average*, the correct exponent is $\frac{1}{4}$. That is, they showed that

$$\frac{1}{X}\int_0^X \left|S_2(r) - \operatorname{Vol} B(\sqrt{r})\right|^2 dr = cX^{\frac{1}{2}} + O(X^{\frac{1}{4}+\epsilon}),$$

and also that

$$S_2(N) - \operatorname{Vol} B(\sqrt{N}) = \Omega(N^{\frac{1}{4}}),$$

giving stronger inclination that the "correct" order of growth is $\frac{1}{4}$ in the exponent.

Connections to modular forms

It turns out that this question is strongly analogous to a similar question on modular forms. We describe that now.

A weight k holomorphic modular form is a holomorphic function f on the upper half-plane $\mathcal{H} = \{x + iy : (x, y) \in \mathbb{R}^2, y > 0\}$, which satisfies a set of "periodicity" conditions

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}(2, \mathbb{Z}),$$

and which is holomorphic at ∞, which translates to f having a Fourier expansion

$$f(z) = \sum_{n \geq 0} a(n) e(nz),$$

(and with a few similar holomorphy conditions that are equivalent to properties of Fourier expansions).

A holomorphic cuspform is a holomorphic modular form with zero constant coefficient in its (various) Fourier expansions:

$$f(z) = \sum_{n \geq 1} a(n) e(nz).$$

The Fourier coefficients of modular forms are well-studied and often correspond to arithmetic quantities. A celebrated theorem of Deligne [Del71] that

$$a(n) \ll n^{\frac{k-1}{2} + \epsilon}.$$

Analogously to the Gauss circle problem, we can consider the sum

$$S_f(X) = \sum_{n \leq X} a(n).$$

By the Deligne bound $a(n) \ll n^{\frac{k-1}{2}+\epsilon}$, we have trivially that

$$S_f(X) \ll X^{\frac{k-1}{2}+1+\epsilon},$$

and if we had square-root cancellation we might expect

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{2}+\epsilon}.$$

Chandrasakharan and Narasimhan [CN62] showed a mean square estimate parallel to that of Hardy and Littlewood for the Gauss circle problem:

$$\frac{1}{X}\int_0^X \left|S_f(t)\right|^2 dt = cX^{k-1+\frac{1}{2}} + O(X^{k-1+\epsilon})$$

Thus as in the Gauss circle problem, we actually conjecture 1/4 as the "correct" exponent.

Conjecture

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

The flavor of Chandrasakharan and Narasimhan's method[1] applies broadly, and actually offers a united approach to both the Gauss circle problem and to the problem of estimating sums of Fourier coefficients of holomorphic cusp forms.

The idea is to start with a Dirichlet series

$$L(s, f) := \sum_{n \geq 1} \frac{a(n)}{\lambda_n^s}$$

that satisfies a functional equation of the form

$$L(s, f)G(s) = L(1 - s, g)G(1 - s),$$

where $G(s)$ is a product of Gamma functions and $L(s, g)$ is a "dual" Dirichlet series

$$L(s, g) = \sum_{n \geq 1} \frac{b(n)}{\mu_n^s}.$$

[1] which perhaps should actually called Landau's method, but Stigler's Law of Eponymy applies here too

Then to study the partial sums $\sum_{n \leq X} a(n)$, CN actually examines a smoothed form (expressed as a Mellin transform) of the Dirichlet series

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, f) X^{s+\delta} \frac{\Gamma(s)}{\Gamma(s+\delta+1)} ds.$$

The idea is that the Gamma functions and basic growth estimates guarantee nice convergence, so one can use Cauchy's residue theorem to shift the line of integration to the left. The "main terms" will be recognized as poles of $L(s)$, and the size of the remainder will be determined by the size of the remaining integral.

It's instructive to examine this in a small bit of additional detail.

Beginning with

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s,f)X^{s+\delta}\frac{\Gamma(s)}{\Gamma(s+\delta+1)}ds,$$

we shift the line of integration to the left to $\operatorname{Re} s = 1-c$, picking up some main terms $\mathrm{MT}(X)$, and leaving the integral

$$\frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} L(s,f)X^{s+\delta}\frac{\Gamma(s)}{\Gamma(s+\delta+1)}ds.$$

Applying the functional equation $L(s,f) = L(1-s,g)G(1-s)/G(s)$ and changing variables $s \mapsto 1-s$, we transform this into

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(s,g)\frac{G(s)}{G(1-s)}X^{s+\delta}\frac{\Gamma(1-s)}{\Gamma(1-s+\delta+1)}ds.$$

If c is large enough that the Dirichlet series $L(s, g)$ converges absolutely, we can swap the order of integration and summation in this integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, g) \frac{G(s)}{G(1-s)} X^{s+\delta} \frac{\Gamma(1-s)}{\Gamma(1-s+\delta+1)} ds.$$

to write this integral as

$$\sum_{n \geq 1} \frac{b(n)}{\mu_n^{\delta+1}} I(\mu_n X),$$

where $I(x)$ is a "smoothed indicator function"

$$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s)}{G(1-s)} X^{s+\delta} \frac{\Gamma(1-s)}{\Gamma(1-s+\delta+1)} ds.$$

CN show that $I(x)$ is well-approximated by a J-Bessel function for large x, and then optimize regions of applying various bounds to the sum to bound the size of the remainder.

Application I: Cusp forms

A holomorphic cuspform $f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$ comes with a Dirichlet series $L(s, f) = \sum_{n \geq 1} a(n) n^{-s}$ satisfying a functional equation of the shape

$$c(f)^s L(s, f) \Gamma(s) = c(f)^{1-s} L(1-s, \widetilde{f}) \Gamma(1-s),$$

and CN's argument gives a bound analogous to the Sierpiński bound for the Gauss circle problem,

$$S_f(N) \ll N^{\frac{k-1}{2} + \frac{1}{3}}.$$

Remarkably, this is the state of the art for this result.[2] There has been no significant[3] improvement since 1962. Morally this should be the same as in the Gauss circle problem, and this result is due for an improvement!

[2] Actually, it's possible to save some fractional log power.

[3] i.e. power savings

Application II: Gauss circle problem

In fact, CN applies also to the Gauss circle problem. The Dirichlet series

$$\zeta_2(s) := \sum_{n \geq 1} \frac{r_2(n)}{n^s}$$

is an "Epstein zeta function" (essentially a sum of a positive definite quadratic form over a lattice), and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta_2(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta_2(1-s).$$

CN's argument in this case matches the Sierpiński bound,

$$S(N) - \operatorname{Vol} B(\sqrt{N}) \ll N^{\frac{1}{3}},$$

(and many of the details are very similar, despite having different colors upon first glance).

Aside on CN

CN wrote a small series of papers based around this technique, with an eye towards studying various estimates of single Dirichlet series. But one can ask about estimates among families of Dirichlet series.

For example, one might want study the number of points in a variety of lattices $\Lambda \subseteq \mathbb{R}^d$ inside a dimensional d-sphere of radius $\sqrt{N}$ and compare point counts among these.

In [LDTT17], Thorne, Taniguchi, and I rework the argument of CN [CN62] to explicitly track the dependence on the lattice Λ, so that the implicit constant depend only on the dimension d. For more general functional equations, we can phrase this as explicitly tracking dependence on all parameters and having implicit constants depend “only on the shape of the functional equation”.

In fact, this can be done to many more of the arguments of CN, but we only carried this out for one particular application so far. There is additional work here to be done uniformizing the results of CN.

Smoothed estimates and the Gauss sphere problem

I spent a long time trying to improve the result to holomorphic cusp forms in various ways. Along with Tom Hulse, Chan Ieong Kuan, and Alex Walker, we found some success studying the Dirichlet series

$$\sum_{n \geq 1} \frac{S_f(n)}{n^s}, \quad \sum_{n \geq 1} \frac{S_2(n)}{n^s}, \quad \text{and} \sum_{n \geq 1} \frac{S_d(n)}{n^s}.$$

Here, we let $S_d(n)$ count the number of lattice points in $\mathbb{Z}^d$ in the ball of radius $\sqrt{n}$, the *Gauss d-sphere problem*.

To non-number theorists, it may be the case that all Dirichlet series look the same. But these are quite weird. Up to now, every Dirichlet series in this talk has secretly been either an L-function or a natural counting zeta function, with beautiful functional equations and excellent properties.

But these series don't have (nice) functional equations. Nonetheless, each of these Dirichlet series has (mostly understandable) meromorphic continuation to the plane and one can still try to apply complex analysis and harmonic analysis to understand their behavior.

At the heart of our analysis are shifted convolution sums in two complex variables,

$$Z(s, w) := \sum_{n,h} \frac{a(n+h)a(n)}{(n+h)^s n^w},$$

which we study using the spectral theory of automorphic forms.

I won't go into the details here, but I've included a large paper trail in which we study these series in the references.

But I will try to give a short impression suggesting that the series

$$\sum_{n \geq 1} \frac{S_f(n)^2}{n^s}$$

is understandable.

We can express the Dirichlet series of interest in terms of the Riemann zeta function $\zeta(s)$, the shifted convolution sum $Z(s,w)$, and the L-function $L(s, f \otimes f)$

$$Z(s,w) = \sum_{n,h} \frac{a(n+h)a(n)}{(n+h)^s n^w}, \quad L(s, f \otimes f) = \sum_n \frac{a(n)^2}{n^s}.$$

One can (essentially combinatorially) decompose the Dirichlet series into

$$\sum_{n \geq 1} \frac{S_f(n)^2}{n^s} = \frac{L(s, f \otimes f)}{\zeta(2s)} + \int_{(\sigma)} \frac{L(s-z, f \otimes f)}{\zeta(2s-2z)} \zeta(z) B(z, s-z) dz$$
$$+ Z(s,0) + \int_{(\sigma)} Z(s-z, 0)\zeta(z) B(z, s-z) dz,$$

where $B(a,b)$ is the Beta function.

This decomposition follows from a Mellin-Barnes type integral identity,

$$\sum_{n,m \geq 1} \frac{a(n)}{(n+m)^s} = \sum_{n,m \geq 1} \int_{(\sigma)} \frac{a(n)}{n^{s-z}} \frac{1}{m^z} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)} dz$$

and a combinatorial game.

From the decomposition

$$\sum_{n\geq 1} \frac{S_f(n)^2}{n^s} = \frac{L(s, f\otimes f)}{\zeta(2s)} + \int_{(\sigma)} \frac{L(s-z, f\otimes f)}{\zeta(2s-2z)}\zeta(z)B(z, s-z)dz$$
$$+ Z(s,0) + \int_{(\sigma)} Z(s-z,0)\zeta(z)B(z,s-z)dz,$$

the moral is that the LHS is understandable as long as $L(s, f\otimes f)$ and $Z(s,w)$ are understandable.

The L-function $L(s, f\otimes f)$ is the Rankin-Selberg convolution of f with itself, and is recognizable as an inner product of the modular form f^2 with an Eisenstein series. The shifted convolution is similarly recognizeable as (the sum of several) inner prodcucts of f^2 with a certain Poincaré series (with auxiliary sum variable w). It's harder to understand, but still comprehendable.

Using these new Dirichlet series, my collaborators and I were able to prove the following smoothed mean square result [HKLDW17b].

Theorem (HKLDW)

$$\sum_{n\geq 1} |S_f(n)|^2 e^{-n/X} = CX^{k-1+\frac{3}{2}} + O(X^{k-1+\frac{1}{2}+\epsilon}).$$

Actually, we prove something a bit mysterious. If g is another weight k cusp form, we show

Theorem (HKLDW)

$$\sum_{n\geq 1} S_f(n)\overline{S_g(n)} e^{-n/X} = C'X^{k-1+\frac{3}{2}} + O(X^{k-1+\frac{1}{2}+\epsilon}).$$

This says something about correlation between the sums $S_f(n)$ and $\overline{S_g(n)}$ which is perhaps non-intuitive. Further, the exponents in the error terms are essentially the best possible, since they correspond to lines of spectral poles (roughly analogous to poles coming from $1/\zeta(s)$).

With this understanding, we went after bounds for individual $S_f(n)$. We chose to do this through a particular short-interval approximation, and in [HKLDW17a] we proved

Theorem (HKLDW II)

$$\frac{1}{X^{\frac{2}{3}}(\log X)^{\frac{1}{6}}} \sum_{|n-X| \leq X^{\frac{2}{3}}(\log X)^{\frac{1}{6}}} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}.$$

This is a sizable (i.e. polynomial) improvement over the previous short-interval estimate, but it's not strong enough to improve individual estimates. With a lot of extra work, we could leverage this to again show

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{3}},$$

i.e. we can perfectly match the result that comes from CN-type methods.

Overall, I've been frustrated and unable to exploit the greater understanding of these exponentially smoothed sums hasn't yielded greater understanding of individual bounds.

Applied to the Gauss circle and d-sphere problems, the story is very similar.

As with the Circle Problem, it is intuitively clear that $S_d(R) \approx \mathrm{Vol}\, B_d(\sqrt{R})$, so the real goal is to understand $|S_d(R) - \mathrm{Vol}\, B_d(\sqrt{R})|$. Through a Gauss-like argument, one can show that

$$S_d(R) - \mathrm{Vol}\, B_d(\sqrt{R}) \ll R^{\frac{d-1}{2}},$$

bounding the error by the surface area.

Conjecture

$$S_d(R) - \mathrm{Vol}\, B_d(\sqrt{R}) \ll R^{\alpha(d)}, \text{where } \alpha(d) = \begin{cases} \frac{1}{4} & d = 2 \\ \frac{d}{2} - 1 & d \geq 3. \end{cases}$$

(Notice the phase shift between dimensions 2 and 3).

In all cases, we have mean-square estimates that lead to the conjectured sizes. For sufficiently high dimensions (i.e. $d \geq 5$), the circle method is enough to prove the conjecture. In 4 dimensions, there is an identity

$$r_4(n) = 8\sigma(n) - 32\sigma(\frac{n}{4}),$$

where $\sigma(n)$ is the sum of divisors of n. This is multiplicative and well behaved, and one can essentially prove the conjecture in dimension 4 using this.

But dimension 2 (the Gauss circle problem) and 3 (the Gauss sphere problem) are very mysterious. In particular, in dimension 3 the best mean square result for decades was due to Jarnik [Jar40], indicating

$$\frac{1}{X}\int_0^X |S_3(r) - \operatorname{Vol} B_3(\sqrt{r})|^2 dr = CX \log X + O(X(\log X)^{\frac{1}{2}}).$$

(Dimension 3 is unique in that the leading term comes with an attached log factor). This is a fractional power of log savings!

The Gauss d-sphere problem is equivalent to studying

$$S_d(R) = \sum_{n \leq R} r_d(n),$$

where $r_d(n) = \#\{z \in \mathbb{Z}^d : x \cdot x = n\}$ is the number of ways of writing n as a sum of d squares.

The Jacobi theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$$

is a (weight $1/2$, non-cuspidal) modular form, and in fact

$$\theta^d(z) = \sum_{n \geq 0} r_d(n) e^{2\pi i n z}$$

is a modular form whose coefficients track $r_d(n)$.

While everything is harder because θ is half-integral weight and non-cuspidal, the techniques still apply.

By studying the (even more non-standard) Dirichlet series $\sum |S_d(n) - \mathrm{Vol}\, B(\sqrt{n})|^2 n^{-s}$, we showed in [HKLDW18a] the following.

Theorem (HKLDW III)

There exists $\lambda > 0$ such that

$$\frac{1}{X}\int_0^X \left|S_3(t) - \mathrm{Vol}\, B_3(\sqrt{t})\right|^2 dt = C' X \log X + CX + O(X^{1-\lambda+\epsilon}).$$

(Alex Walker computed that we can take $\lambda = \frac{1}{5}$).

Further, if we consider smoothed approximations, we can show that

Theorem (HKLDW III)

$$\begin{aligned}\sum_{n\geq 1} \left|S_d(n) - \mathrm{Vol}\, B_d(\sqrt{n})\right|^2 e^{-n/X} \\ = \delta_{[d=3]} C' X^2 \log X + C_d X^{d-1} \\ + \delta_{[d=4]} C_4 X^{\frac{5}{2}} + C_d'' X^{d-2} + O(X^{d-\frac{5}{2}+\epsilon}).\end{aligned}$$

This is very nearly the Laplace transform. In terms of the Laplace transform, we can phrase our result as

$$\int_0^\infty \left|S_d(t) - \operatorname{Vol} B_d(\sqrt{t})\right|^2 e^{-t/X} dt = \delta_{[d=3]} C' X^2 \log X + C_d X^{d-1} + \delta_{[d=4]} C_4 X^{\frac{5}{2}} + C_d'' X^{d-2} + O(X^{d-\frac{5}{2}+\epsilon})$$

(with slightly different, but explicit, constants). In fact, we can continue outputting additional terms down to $O(X^{\frac{d-1}{2}+\epsilon})$, but then another line of spectral poles prevents further understanding.

Unfortunately, I am again stumped. I had expected that improving results for the Laplace transform, short-interval estimates, or sharp second moments would translate into improved results for individual bounds — but I haven't been able to prove these. Instead, I've proved that there is a wall of highly oscillatory terms coming from spectral poles that are completely mysterious to me.

Lattice points on hyperboloids

The d-dimensional Gauss Sphere Problem concerns counting

$$\#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 \leq R\} = \sum_{m \leq R} r_d(m).$$

Suppose instead we want to count the number of lattice points on the one-sheeted hyperboloid $\mathcal{H}_{d,h}$ for some positive integer h,

$$\#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d^2 + h\}$$

that are contained within the (dimension d) ball $B(\sqrt{R})$ This is equivalent to counting

$$\sum_{2m^2+h \leq R} r_{d-1}(m^2 + h),$$

which looks very similar to the Gauss $d-1$ Sphere Problem sum, except constrained along a surface.

In many dimensions, the circle method should be able to determine a main term with some logarithmic savings, with better savings occurring for very high dimension.

The two dimensional case is now uninteresting, but the three-dimensional case is again very enigmatic. When h is a square, it is easy to come up with a heuristic. Consider

$$X^2 + Y^2 = Z^2 + h^2.$$

Then setting $X = Z$, $Y = h$ gives $\sqrt{R}$ trivial terms. It's natural to ask: Are these most of the solutions, or are we missing many more?

Oh and Shaw [OS11] recently showed that when h is a square, the total number of solutions is

$$C\sqrt{R}\log R + O(R^{\frac{1}{2}}(\log R)^{\frac{3}{4}}).$$

So we see that most solutions are nontrivial.

We return now to the standard hyperboloid $X^2 + Y^2 = Z^2 + h$, when h is not necessarily a square. The key idea to my approach is that these solutions can also be retrieved from a modular form, namely $V(z) = \theta^2(z)\overline{\theta(z)}$.

In particular, the hth Fourier coefficient of $V(z)$ is given by

$$\sum_{m \in \mathbb{Z}} r_2(m^2 + h) e^{-(2m^2+h)\pi y},$$

which is an exponentially weighted version of the sum we want to understand.

Using analogous techniques to study the shifted convolution sum briefly touched on earlier, we can recover the Dirichlet series

$$\sum_{m \in \mathbb{Z}} \frac{r_2(m^2 + h)}{(2m^2 + h)^s}.$$

Further, it's possible to attain the meromorphic continuation for the Dirichlet series

$$\sum_{m\in\mathbb{Z}} \frac{r_{d-1}(m^2+h)}{(2m^2+h)^s}.$$

and it is possible to use this Dirichlet series to prove a variety of results.

Of particular interest is the following

Theorem (DLD)

The number of integer lattice points on the hyperboid $\mathcal{H}_{3,h}$ and within the ball of radius $\sqrt{R}$ centered at the origin is

$$\delta_{[h=a^2]} C' R^{\frac{1}{2}} \log R + CR^{\frac{1}{2}} + O(R^{\frac{1}{2}-\lambda+\epsilon})$$

for a positive $\lambda > 0$ (which might be $1/44$).

As a corollary, note that when h is not a square, a positive proportion of solutions are trivial solutions!

Takeaways

An underlying theme o this talk is that the Gauss circle problem is analogous to several problems concerning the Fourier coefficients of modular forms. Often, techniques that work in one domain apply in some form to the other, but not all of these domains are equally studied.

In particular, much of the work in the classical Gauss circle problem is due to the circle method and harmonic analysis. But most of the results I've described in this talk end with certain techniques and results from the spectral theory of automorphic forms and complex analysis.

I suspect that with more harmonic analysis, it would be possible to improve the various results in this talk. Perhaps we could even improve the individual bound for $S_f(X) = \sum_{n \leq X} a(n)$!

Takeaways

An underlying theme o this talk is that the Gauss circle problem is analogous to several problems concerning the Fourier coefficients of modular forms. Often, techniques that work in one domain apply in some form to the other, but not all of these domains are equally studied.

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Or maybe not — I'm not sure!

Thank you very much.

Please note that these slides (and references for the cited works) are (or will soon be) available on my website (davidlowryduda.com).

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