Arithmetic Problems with Dirichlet Series having Lines of Poles

Proving $\Omega_{\pm}$ results

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Maine-Quebec Number Theory Seminar
October 2019

ICERM and Brown University

Acknowledgements

Many of the ideas in this talk comes were inspired by past and present work with a few collaborators, including Tom Hulse (Boston College), Chan Ieong Kuan (Sun-Yat Sen University), and Alex Walker (Rutgers).

Other parts come from conversations with many and numerous people who have been kind enough to lend me an ear and a hand.

A classical pattern

Often we have an arithmetic sequence we want to understand, and we try to understand it through it's Dirichlet series. A few examples are

- (Epstein zeta function) $\zeta_2(s) = \sum_{n \geq 1} \frac{r_2(n)}{n^s}$, where $r_2(n)$ is the number of ways of representing n as a sum of 2 squares.
- (modular L-function) $L(s, f) = \sum_{n \geq 1} \frac{a(n)}{n^{s+\frac{k-1}{2}}}$, where $f = \sum a(n)e(nz)$ is a weight k cuspidal modular form.

Or perhaps even

- (logarithic derivative of the zeta function) $\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$, where $\Lambda(n)$ is the von Mangoldt function.

Each of these are related to classical problems in analytic number theory.

Given each Dirichlet series, it is very natural to extract information about the partial sums of the coefficients.

To do so, we use the analytic properties about the Dirichlet series and use contour integrals (inverse Mellin transforms)

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{D(s)}{X^s}ds.$$

Other sorts of integral transforms give access to a rich variety of other information about the sequences.

From $\zeta_2(s)$, we can study $S_2(R) = \sum_{n \le R} r_2(n)$, which counts the number of integer lattice points in a circle of radius R. This is known as the *Gauss circle problem*.$\zeta_2(s)$ has a pole at $s = 1$, leading to

$$S_2(R) = \sum_{n \le R} r_2(n) = \pi R + E_2(R)$$

for some error term $E_2(R)$.

From $L(s, f)$, we can study $S_f(X) = \sum_{n \le X} a(n)$, the average order of coefficients of the cusp form f. $L(s, f)$ has no poles, and so we get merely that

$$S_f(X) = \sum_{n \le X} a(n) = E_f(X),$$

some sort of error term.

From $\zeta'(s)/\zeta(s)$, we can study $\psi(X) = \sum_{n \le X} \Lambda(n)$, (the Chebyshev function). This has a pole at $s = 1$ and at each zero of the zeta function, leading to the *explicit formula* (used to prove the prime number theorem)

$$\psi(X) = \sum_{n \le X} \Lambda(n) = X - \sum_{\rho} \frac{X^\rho}{\rho} + E_\Lambda(X).$$

Understanding Error Terms

The question then becomes: how do we understand the error terms? For $S_2(R)$ and $S_f(n)$, this is often done by studying the Dirichlet series of squared coefficients:

$$\sum_{n\geq 1} \frac{r_2(n)^2}{n^s}, \quad \sum_{n\geq 1} \frac{a(n)^2}{n^{s+k}}.$$

Alternately, over the last few years my collaborators and I have introduced and studied the Dirichlet series formed of the partial sums themselves:

$$\sum_{n\geq 1} \frac{S_2(n)^2}{n^s}, \quad \sum_{n\geq 1} \frac{E_2(n)^2}{n^s}, \quad \sum_{n\geq 1} \frac{S_f(n)^2}{n^s}$$

We show that each of these series has meromorphic continuation to $\mathbb{C}$. Each have poles at $s = 1$ (when normalized). But like ζ'/ζ, each have infinitely many poles at locations we don't quite understand, and with residues we don't understand.

The poles

Let's give an idea of where these poles come from in the case of the Gauss Circle problem.

There is a decomposition of the shape

$$\sum_{n\geq 1} \frac{S_2(n)^2}{n^s} \approx \sum_{n\geq 1} \frac{r_2(n)^2}{n^s} + \sum_{m,n\geq 1} \frac{r_2(n)r_2(n+m)}{(n+m)^s} + \text{integral transform.}$$

The first sum is understandable — in fact it's exactly

$$\sum_{n\geq 1} \frac{r_2(n)^2}{n^s} = \frac{16\zeta(s)^2 L(s,\chi_{-4})^2}{(1+2^{-s})\zeta(2s)}.$$

But the second is harder to understand.

Spectral decomposition and spectral poles

To understand the second term, we phrase the problem in terms of modular forms and use a spectral decomposition.

In this case, $\theta(z)^2 = 1 + \sum_{n \geq 1} r_2(n) e(nz)$ is a weight 1 modular form on $\Gamma_0(4)$, and using its spectral decomposition, one can show that

$$\sum_{n,m \geq 1} \frac{r_2(n) r_2(n+m)}{(n+m)^2 m^w} = \sum_j (\text{Maass data}) G(s, it_j) + \sum_{\mathfrak{a}} \int (\text{Eisenstein data}),$$

a decomposition in terms of data corresponding to Maass waveforms and integrals against Eisenstein series associated to the cusps of $\Gamma_0(4)$. I've omitted almost every detail except for $G(s, it_j)$, which is the ratio of Gamma functions

$$\frac{\Gamma(s - \frac{1}{2} + it_j)\Gamma(s - \frac{1}{2} - it_j)}{\Gamma(s)^2},$$

which have lots and lots of poles.

Confusing asymptotics

With this, we show

$$\sum_{n\geq 1}\frac{E_2(n)^2}{n^s}$$

has poles at $s = 3/2$, $s = 1$, $s = 1/2$, and at $s = \frac{1}{2} \pm it_j$ for each t_j coming from a Maass form.

One could use this to show, for example,

$$\sum_{n\geq 1} E_2(n)^2 e^{-n/X} = cX^{3/2} + c'X + c''\sqrt{X} + \sum_{\pm it_j} c_{\pm t_j} X^{\frac{1}{2}\pm it_j} + O(X^{1/4+\epsilon})$$

But we know almost nothing about the residual coefficients $c_{\pm t_j}$ or the distribution of the it_j themselves, except that they are $O(X^{1/2+\epsilon})$ overall.

Is it possible that there is tremendous cancellation all the time, so the spectral sum is much smaller than we expect all the time? Or is the term of size $\Omega(X^{1/2})$? How do we understand these terms?

1920s tech

Until this week, I thought I was going to present a variety of other arithmetic phenomena that have spectral poles giving what looks like Ω terms, but which I didn't understand.

But this week, I happened across the work of Ingham in the 1920s, and I learned an interesting technique.

A theorem

Theorem

Suppose $D(s) = \sum_n a(n)n^{-s}$ *is a Dirichlet series that converges absolutely somewhere, and which has meromorphic continuation to* $\mathbb{C}$*. Further, suppose* D *has at least one non-real pole, and let* $m := \sup\{\mathrm{Re} s : s \text{ a non-real pole}\}$.

Let $S(x) = \sum_{n \leq X} a(n)$.

Finally, let $\mathrm{MT}(X)$ *be the sum of the residues of the real poles of* $D(s)X^s/s$ *with real part* $\geq m$.

Then

$$S(X) - \mathrm{MT}(X) = \Omega_{\pm}(X^{m-\epsilon})$$

for every $\epsilon > 0$.

Here, $f(x) = \Omega_{\pm}(g(x))$ means that there is a constant $c > 0$ such that $\limsup f(x)/g(x) > c$ and $\liminf f(x)/g(x) < c$ — or rather that $f(x)$ is at least as large as g both positively and negatively.

Thus, for example, we have that

$$\sum_{n \leq X} E_2(n)^2 - (cX^{3/2} + c'X + c''\sqrt{X}) = \Omega_{\pm}(X^{1/2-\epsilon}).$$

Or, if Θ is the supremum of the real parts of the non-trivial zeros of the zeta function, then

$$\sum_{n \leq X} \Lambda(n) - X = \Omega_{\pm}(X^{\Theta-\epsilon}).$$

(Presumably $\Theta = 1/2$.)

Main idea of the proof

Let $D(s) = \sum_{n \geq 1} a(n) n^{-s}$, and suppose that $D(s)$ converges absolutely for $\mathrm{Re} s > \sigma_0$ (i.e. that σ_0 is the abscissa of convergence) and has meromorphic continuation to $\mathbb{C}$.

There is a theorem of Landau that says that if $a(n) > 0$ for all sufficiently large n, then σ_0 is a singularity of $D(s)$.

This can be phrased through the Mellin transform as well.

Dirichlet integrals

That is, suppose $a(x)$ is an integrable function, and

$$F(s) = \int_1^\infty \frac{a(x)}{x^s} dx$$

converges at s_0. Then

1. (many results of Dirichlet series hold) for all s with $\mathrm{Re} s > \mathrm{Re} s_0$, $F(s)$ converges.
2. (Landau's Theorem) if $a(x) > 0$ for all sufficiently large x, then $\sigma_0 = \mathrm{Re} s_0$ is a singularity of $F(s)$.

And more generally, essentially the whole theory of Dirichlet series applies to these Dirichlet integrals as well. (And was being actively studied in the 1920s).

Proof of $\Omega_\pm$ theorem

With this, we can now describe how to prove the $\Omega_\pm$ theorem.

Recall we have $D(s) = \sum a(n) n^{-s}$; $S(X) = \sum_{n \le X} a(n)$; m is the supremum of the real parts of non-real poles; and $\mathrm{MT}(X)$ is the sum of the residues of $D(s)X^s/s$ with $\mathrm{s} \ge m$.

Now suppose that for some $k < m$, we have that $S(x) - \mathrm{MT}(x) - x^k > 0$ for all sufficiently large x. Consider

$$F(s) = \int_1^\infty \frac{S(x) - \mathrm{MT}(x) - x^k}{x^{s+1}} dx.$$

$F(s)$ is essentially the Dirichlet series $D(s)$ with the leading real poles removed, minus $1/(s-k)$. In particular, $F(s)$ converges for $\mathrm{Re} s > m$, but has at least one non-real singularity near $\mathrm{Re} s = m$. But by Landau's Theorem, the first singularity needs to be real contradiction! (a similar argument applies in the always negative case).

Thank you very much.

Please note that these slides (and references for the cited works) are available on my website (davidlowryduda.com).

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