Counting Lattice Points on Spheres and Hyperboloids

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One-Sheeted Hyperboloids

A one-sheeted hyperboloid $\mathcal{H}_d(h)$ is described by the equation

$$X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h,$$

where $h > 0$. Of particular interest is the 3-dimensional hyperboloid $\mathcal{H}_3(h)$, which is given by

$$X^2 + Y^2 = Z^2 + h.$$

One might ask how many lattice points are on these hyperboloids? *(Answer: infinitely many).*

How many lattice points are on these hyperboloids, and are not too large? This is our guiding question.

Counting points on one-sheeted hyperboloids

Let $h \in \mathbb{Z}^+$. How many lattice points are on the surface of the hyperboloid $\mathcal{H}_d(h)$ *and* contained within the ball $B(\sqrt{R})$ of radius $\sqrt{R}$?

This question is similar in flavor to the Generalized Gauss Circle Problem of counting the number of lattice points within the d-dimensional ball $B(\sqrt{R})$, except that in our problem we impose an additional constraint.

As in the Generalized Gauss Circle Problem, the Hardy-Littlewood circle method can be applied for sufficiently large dimension. But smaller dimensions are more challenging. And the 3-dimensional case is by far the most enigmatic.

Oh and Shah [OS11] recently showed using ergodic techniques to show that the number of lattice points on

$$X^2 + Y^2 = Z^2 + h^2$$

and within $B(\sqrt{R})$ is

$$C\sqrt{R}\log R + O(R^{\frac{1}{2}}(\log R)^{\frac{3}{4}})$$

for an explicit constant C depending on h. Producing a better asymptotic and understanding the case when h is not a square has proved challenging.

A New Approach

Note that if $X^2 + Y^2 = Z^2 + h$, then the condition of being contained in $B(\sqrt{R})$ is the same as

$$(X^2 + Y^2) + Z^2 \leq R \iff 2Z^2 + h \leq R.$$

By considering each possible value of $Z^2 + h$ separately, we see that the number of points on $\mathcal{H}_3$ and within $B(\sqrt{R})$ is given by

$$\sum_{2n^2+h \leq R} r_2(n^2 + h) \approx \frac{1}{2} \sum_{2n+h \leq R} r_2(n + h) r_1(n),$$

where $r_k(n)$ is the number of ways of representing n as a sum of k squares.

These sums appear as Perron-type integrals of the Dirichlet series

$$\sum_{n \geq 0} \frac{r_2(n^2 + h)}{(n^2 + h)^s} \approx \frac{1}{2} \sum_{n \geq 0} \frac{r_2(n + h) r_1(n)}{(n + h)^s}.$$

If we can understand this Dirichlet series, we can produce asymptotics.

Convolution Sums

This Dirichlet series is a shifted convolution sum associated to (a product of two) modular forms. Let

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = 1 + \sum_{n \geq 1} r_1(n) e^{2\pi i n z}$$

be the standard Jacobi theta function, a modular form of weight $1/2$ on $\Gamma_0(4)$. (Note also that $\theta^d(z) = 1 + \sum_{n \geq 1} r_d(n) e^{2\pi i n z}$).

The relevant modular form for $\mathcal{H}_3$ is $V(z) = \theta^2(z)\overline{\theta(z)}$. (And for $\mathcal{H}_d$, it's $\theta^{d-1}(z)\overline{\theta(z)}$).

In particular, the hth Fourier coefficient of $V(z)$ is given by

$$\sum_{n \in \mathbb{Z}} r_2(n^2 + h) e^{-(2n^2+h)\pi y},$$

which is an exponentially weighted version of the sum we want to understand.

Convolutions through Poincaré Series

Let $P_h^k(z,s)$ be a weight k real-analytic Poincaré series,

$$P_h^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s e^{2\pi i h \gamma z} J(\gamma, z)^{-k}$$

where $J(\gamma,z) = j(\gamma,z)/|j(\gamma,z)|$ and $j(\gamma,z)$ is our multiplier.(Which is different for full and half integral weight).

If $h = 0$, then this gives the weight k real-analytic Eisenstein series

$$E_\infty^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s J(\gamma,z)^k.$$

Both P_h^k and E_∞^k transform like weight k modular forms (in z), and are meromorphic functions in s.

Recognize the Dirichlet Series

The Petersson inner product of $P_h^{\frac{1}{2}}(z,s)$ against V gives

$$\langle P_h^{\frac{1}{2}}(z,s), V(z)\rangle = \iint_{\Gamma_0(4)\backslash\mathcal{H}} P_h^{\frac{1}{2}}(z,s)\overline{\theta^2(z)}\theta(z)y^{\frac{3}{4}}\frac{dxdy}{y^2}$$
$$= \frac{\Gamma(s-\frac{1}{4})}{(2\pi)^{s-\frac{1}{4}}}\sum_{m\in\mathbb{Z}}\frac{r_2(m^2+h)}{(m^2+h)^{s-\frac{1}{4}}},$$

which is our Dirichlet series (and an easily understood analytic factor). We have reduced the task to understanding the inner product on the left.

This follows a general plan for constructing shifted convolution sums. If f and g are two modular forms with coefficients $a(n)$ and $b(n)$ respectively, then roughly speaking

$$\langle P_h(z,s), f(z)\overline{g(z)}\rangle \approx \frac{\Gamma(s)}{(2\pi)^s}\sum\frac{\overline{a(m+h)}b(m)}{(m+h)^s}.$$

Spectrally Expand the Poincaré Series

The Poincaré series has a spectral expansion of the form

$$\begin{aligned} P_h^k(z,s) = &\sum_j \langle P_h^k, \mu_j\rangle \mu_j(z) + \sum_{\frac{1}{2}\leq \ell \leq k} \sum_j \langle P_h^k, \mu_{j,\ell}\rangle \mu_{j,\ell}(z) \\ &+ \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k\rangle R_{\mathfrak{a}}^k \\ &+ \sum_{\mathfrak{a}} \int_{(1/2)} \langle P_h^k, E_{\mathfrak{a}}^k(\cdot, u)\rangle E_{\mathfrak{a}}^k(z,u) du. \end{aligned}$$

The first line is the discrete spectrum. $\mu_j(z)$ denotes a Maass form of weight k (and $\mu_{j,\ell}(z)$ denotes Maass forms that come from products of lower weight Maass forms and holomorphic cusp forms). The second line is the residual spectrum, and only appears when k is a half-integer. The third line is the continuous spectrum, and each $E_{\mathfrak{a}}^k$ is a weight k Eisenstein series associated to the cusp $\mathfrak{a}$.

Substitute into the Inner Product

The plan is to substitute the spectral expansion into the inner product $\langle P_h^{\frac{1}{2}}, V \rangle$ and understand the behavior as a meromorphic function of s. In the analogous case when k is a full integer and when V is f or $f\overline{g}$ (where f and g are cusp forms), this is fairly straightforward, and can be used towards many different goals.

But in this case, there are two major obstructions:

1. V is is non-cuspidal. This means that it's not possible to naively substitute the spectral expansion and consider each part of the sum separately, since convergence is not guaranteed.
2. $V = \theta^2\overline{\theta}$ comes from half-integral weight objects, which generically behave worse than full-integral weight objects.

But one can get around these obstructions.

Instead of V, one can consider $\widetilde{V} = V - V_0 - V_\infty - V_{1/2}$, where the three functions V_0, V_∞, and $V_{1/2}$ cancel the growth of V at the three cusps of $\Gamma_0(4)$.

For $V = \theta^{2k+1}(z)\overline{\theta(z)}$ (which corresponds to dimension $2k+2$), it turns out that $\widetilde{V} = V - E^k_\infty(z, \frac{k+1}{2}) - E^k_0(z, \frac{k+1}{2})$ suffices... except in dimension 3 (when $k = 1/2$), where this happens to lie on a pole of the half-integral weight Eisenstein series.But then one can use just the constant term of the Laurent expansions of the Eisenstein series, expanded at that pole.

Then one replaces each V in the proof outline with $\widetilde{V}$, which smoothes away many of the difficulties.

The other major obstruction comes from half-integral weight objects, especially in the spectral expansion. These are serious difficulties. But in a few places, I used one particular trick, which I don't think is well-known enough.

The theta function $\theta(z)$ is a residue of the half-integral weight real-analytic Eisenstein series,

$$\mathrm{Res}_{w=3/4} E_\infty^{1/2}(z, w) = c y^{1/4} \theta(z).$$

Thus a generic term in the discrete spectrum can be thought of as

$$\langle \mu_j, \theta^2(z) \overline{\theta(z)} y^{3/4} \rangle \approx \mathrm{Res} \langle \mu_j, \theta^2(z) \overline{E_\infty^{1/2}(z, w)} y^{1/2} \rangle.$$

The object on the right is essentially another name for the Rankin-Selberg L-function $\sum r_2(n) \rho_j(n) / n^s$ (where $\rho_j(n)$ are the coefficients of μ_j).

Additional details can be seen in Chapter 5 of my thesis [LD17].And shortly a preprint on the arXiv...

The end result is that we get a complete meromorphic continuation to $\mathbb{C}$ of the series

$$\sum_{m\in\mathbb{Z}} \frac{r_{d-1}(m^2+h)}{(m^2+h)^s}.$$

and it is possible to use this Dirichlet series to prove a variety of results.

Theorem

The number of integer lattice points on the hyperboid $\mathcal{H}_3$ and within the ball of radius $\sqrt{R}$ centered at the origin is

$$\sum_{2n^2+h\leq R} r_2(n^2+h) = \delta_{[h=a^2]} C' R^{\frac{1}{2}} \log R + CR^{\frac{1}{2}} + O(R^{\frac{1}{2}-\frac{1}{44}+\epsilon}).$$

More generally, for $\mathcal{H}_d(h)$ and $d \geq 4$, we have

$$\sum_{2n^2+h\leq R} r_d(n^2+h) = CR^{\frac{d-2}{2}} + O(R^{\frac{d-2}{2}-\lambda+\epsilon})$$

for a computable $\lambda = \lambda(d) > 0$.

Thank you very much.

Please note that these slides (and references for the cited works) are available on my website (davidlowryduda.com).

References i

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