Counting Points on One-Sheeted Hyperboloids

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One-Sheeted Hyperboloids

A one-sheeted hyperboloid $\mathcal{H}_d(h)$ is described by the equation

$$X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h,$$

where $h > 0$. Of particular interest is the 3-dimensional hyperboloid $\mathcal{H}_3(h)$, which is given by

$$X^2 + Y^2 = Z^2 + h.$$

One might ask how many lattice points are on these hyperboloids? *(Answer: infinitely many).*

How many lattice points are on these hyperboloids, and are not too large? This is our guiding question.

Counting points on one-sheeted hyperboloids

Let $h \in \mathbb{Z}^+$. How many lattice points are on the surface of the hyperboloid $\mathcal{H}_d(h)$ *and* contained within the ball $B(\sqrt{R})$ of radius $\sqrt{R}$?

This question is similar in flavor to the Generalized Gauss Circle Problem of counting the number of lattice points within the d-dimensional ball $B(\sqrt{R})$, except that in our problem we impose an additional constraint.

As in the Generalized Gauss Circle Problem, the Hardy-Littlewood circle method can be applied for sufficiently large dimension. But smaller dimensions are more challenging. And the 3-dimensional case is by far the most enigmatic.

Oh and Shah [OS11] recently showed using ergodic techniques to show that the number of lattice points on

$$X^2 + Y^2 = Z^2 + h^2$$

and within $B(\sqrt{R})$ is

$$C\sqrt{R}\log R + O(R^{\frac{1}{2}}(\log R)^{\frac{3}{4}})$$

for an explicit constant C depending on h. Producing a better asymptotic and understanding the case when h is not a square has proved challenging.

A New Approach

Note that if $X^2 + Y^2 = Z^2 + h$, then the condition of being contained in $B(\sqrt{R})$ is the same as

$$(X^2 + Y^2) + Z^2 \leq R \iff 2Z^2 + h \leq R.$$

By considering each possible value of $Z^2 + h$ separately, we see that the number of points on $\mathcal{H}_3$ and within $B(\sqrt{R})$ is given by

$$\sum_{2n^2+h \leq R} r_2(n^2 + h) \approx \frac{1}{2} \sum_{2n+h \leq R} r_2(n + h) r_1(n),$$

where $r_k(n)$ is the number of ways of representing n as a sum of k squares.

These sums appear as Perron-type integrals of the Dirichlet series

$$\sum_{n \geq 0} \frac{r_2(n^2 + h)}{(n^2 + h)^s} \approx \frac{1}{2} \sum_{n \geq 0} \frac{r_2(n + h) r_1(n)}{(n + h)^s}.$$

If we can understand this Dirichlet series, we can produce asymptotics.

Modular Forms

The key insight is that this Dirichlet series can also be retrieved from a modular form. Let

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = 1 + \sum_{n \geq 1} r_1(n) e^{2\pi i n z}$$

be the standard Jacobi theta function, a modular form of weight $1/2$ on $\Gamma_0(4)$. (Note also that $\theta^d(z) = 1 + \sum_{n \geq 1} r_d(n) e^{2\pi i n z}$).

The relevant modular form for $\mathcal{H}_3$ is $V(z) = \theta^2(z)\overline{\theta(z)}$. (And for $\mathcal{H}_d$, it's $\theta^{d-1}(z)\overline{\theta(z)}$).

In particular, the hth Fourier coefficient of $V(z)$ is given by

$$\sum_{n \in \mathbb{Z}} r_2(n^2 + h) e^{-(2n^2+h)\pi y},$$

which is an exponentially weighted version of the sum we want to understand.

Methodology

Let $P_h^k(z,s)$ be a weight k Poincaré series,

$$P_h^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s e^{2\pi i h \gamma z} J(\gamma, z)^{-k}$$

where $J(\gamma,z) = j(\gamma,z)/|j(\gamma,z)|$ and $j(\gamma,z)$ is our multiplier.

If $h = 0$, then this gives the weight k real-analytic Eisenstein series

$$E_\infty^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s J(\gamma, z)^k.$$

Both P_h^k and E_∞^k transform like weight k modular forms (in z), and are meromorphic functions in s.

On Multipliers for Full-Integral vs Half-Integral k

When k is an integer and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, we take $j(\gamma, z) = (cz + d)$. When k is a half-integer, we must be a bit more careful to ensure a well-defined multipler system, and we take

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2},$$

where ε_d is 1 if $d \equiv 1 \bmod 4$ and is i if $d \equiv 3 \bmod 4$ and $\left(\frac{c}{d}\right)$ is a Jacobi symbol.

Half-integral weight modular forms lead to a few more complications than full-integral weight forms, but I omit most of these details from this talk.

Recognize the Dirichlet Series

The Petersson inner product of $P_h^{\frac{1}{2}}(z,s)$ against V gives

$$\begin{aligned}\langle P_h^{\frac{1}{2}}(z,s), V(z)\rangle &= \iint_{\Gamma_0(4)\backslash\mathcal{H}} P_h^{\frac{1}{2}}(z,s)\overline{\theta^2(z)}\theta(z)y^{\frac{3}{4}}\frac{dxdy}{y^2}\\ &= \frac{\Gamma(s-\frac{1}{4})}{(2\pi)^{s-\frac{1}{4}}}\sum_{m\in\mathbb{Z}}\frac{r_2(m^2+h)}{(m^2+h)^{s-\frac{1}{4}}},\end{aligned}$$

which is our Dirichlet series (and an easily understood analytic factor). We have reduced the task to understanding the inner product on the left.

I note that this follows a general plan for constructing shifted convolution sums. If f and g are two modular forms with coefficients $a(n)$ and $b(n)$ respectively, then roughly speaking

$$\langle P_h(z,s), f(z)\overline{g(z)}\rangle \approx \frac{\Gamma(s)}{(2\pi)^s}\sum\frac{\overline{a(m+h)}b(m)}{(m+h)^s}.$$

Spectrally Expand the Poincaré Series

The Poincaré series has a spectral expansion of the form

$$P_h^k(z,s) = \sum_j \langle P_h^k, \mu_j \rangle \mu_j(z) + \sum_{\frac{1}{2} \leq \ell \leq k} \sum_j \langle P_h^k, \mu_{j,\ell} \rangle \mu_{j,\ell}(z)$$
$$+ \sum_{\mathfrak{a}} \langle P_h^k, R_{\mathfrak{a}}^k \rangle R_{\mathfrak{a}}^k$$
$$+ \sum_{\mathfrak{a}} \int_{(1/2)} \langle P_h^k, E_{\mathfrak{a}}^k(\cdot, u) \rangle E_{\mathfrak{a}}^k(z,u) du.$$

The first line is the discrete spectrum, $\mu_j(z)$ denotes Maass forms of weight k (and $\mu_{j,\ell}(z)$ denotes Maass forms that come from products of lower weight Maass forms and holomorphic cusp forms). The second line is the residual spectrum, and only appears when k is a half-integer. The third line is the continuous spectrum, and each $E_{\mathfrak{a}}^k$ is a weight k Eisenstein series associated to the cusp $\mathfrak{a}$.

Substitute into the Inner Product

The plan is to substitute the spectral expansion into the inner product $\langle P_h^{\frac{1}{2}}, V \rangle$ and understand the behavior as a meromorphic function of s. In the analogous case when k is a full integer and when V is f or $f\overline{g}$ (where f and g are cusp forms), this is fairly straightforward, and can be used towards many different goals.

But in this case, there are two major obstructions:

1. V is is non-cuspidal. This means that it's not possible to naively substitute the spectral expansion and consider each part of the sum separately, since convergence is not guaranteed.
2. $V = \theta^2\overline{\theta}$ comes from half-integral weight objects, which generically behave worse than full-integral weight objects.

But one can get around these obstructions.

Instead of V, one can consider $\widetilde{V} = V - V_0 - V_\infty - V_{1/2}$, where the three functions V_0, V_∞, and $V_{1/2}$ cancel the growth of V at the three cusps of $\Gamma_0(4)$.

For $V = \theta^{2k+1}(z)\overline{\theta(z)}$ (which corresponds to dimension $2k+2$), it turns out that $\widetilde{V} = V - E_\infty^k(z, \frac{k+1}{2}) - E_0^k(z, \frac{k+1}{2})$ suffices...except in dimension 3 (when $k = 1/2$), where this happens to lie on a pole of the half-integral weight Eisenstein series. But then it suffices to only subtract just the constant term of the Laurent expansions of the Eisenstein series, expanded at that pole.

Then one replaces each V in the proof outline with $\widetilde{V}$, which smoothes away many of the difficulties.

The other major obstruction comes from half-integral weight objects, especially in the spectral expansion. To some extent, these are serious difficulties. But in a few places, I used one particular trick.

The theta function $\theta(z)$ is a residue of the half-integral weight real-analytic Eisenstein series,

$$\mathrm{Res}_{w=3/4} E_\infty^{1/2}(z, w) = c y^{1/4} \theta(z).$$

Thus for example in the discrete spectrum, we can think of

$$\langle \mu_j, \theta^2(z) \overline{\theta(z)} y^{3/4} \rangle \approx \mathrm{Res} \langle \mu_j, \theta^2(z) \overline{E_\infty^{1/2}(z, w)} y^{1/2} \rangle,$$

and the object on the right is essentially another name for the Rankin-Selberg L-function $\sum r_2(n) \rho_j(n) / n^s$ (where $\rho_j(n)$ are the coefficients of μ_j).

Additional details can be seen in Chapter 5 of my thesis [LD17].

The end result is that we get a complete meromorphic continuation to $\mathbb{C}$ of the series

$$\sum_{m \in \mathbb{Z}} \frac{r_{d-1}(m^2+h)}{(m^2+h)^s}.$$

and it is possible to use this Dirichlet series to prove a variety of results.

Theorem

The number of integer lattice points on the hyperboid $\mathcal{H}_3$ and within the ball of radius $\sqrt{R}$ centered at the origin is

$$\sum_{2n^2+h \leq R} r_2(n^2+h) = \delta_{[h=a^2]} C' R^{\frac{1}{2}} \log R + CR^{\frac{1}{2}} + O(R^{\frac{1}{2}-\frac{1}{44}+\epsilon}).$$

More generally, for $\mathcal{H}_d(h)$ and $d \geq 4$, we have

$$\sum_{2n^2+h \leq R} r_d(n^2+h) = CR^{\frac{d-2}{2}} + O(R^{\frac{d-2}{2}-\lambda+\epsilon})$$

for a computable $\lambda = \lambda(d) > 0$.

Thank you very much.

References i

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