A SHORT NOTE ON GAPS BETWEEN POWERS OF CONSECUTIVE PRIMES

DAVID LOWRY-DUDA

ABSTRACT. Let $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$. In this short note, we show that $\liminf_{n\to\infty} p_{n+1}^{\alpha} - p_n^{\alpha} = 0$, where p_n is the nth prime. This notes an improvement over results of Sándor and gives additional evidence towards a conjecture of Andrica. This follows directly from recent results on prime pairs from Maynard, Tao, Zhang.

1. Introduction

The primary purpose of this note is to collect a few hitherto unnoticed or unpublished results concerning gaps between powers of consecutive primes. The study of gaps between primes has attracted many mathematicians and led to many deep realizations in number theory. The literature is full of conjectures, both open and closed, concerning the nature of primes.

In a series of stunning developments, Zhang, Maynard, and Tao [May15, Zha14] made the first major progress towards proving the prime k-tuple conjecture, and successfully proved the existence of infinitely many pairs of primes differing by a fixed number. As of now, the best known result is due to the massive collaborative Polymath8 project [Pol14], which showed that there are infinitely many pairs of primes of the form $p, p + 246$. In the excellent expository article [Gra15], Granville describes the history and ideas leading to this breakthrough, and also discusses some of the potential impact of the results. This note should be thought of as a few more results following from the ideas of Zhang, Maynard, Tao, and the Polymath8 project.

Throughout, p_n will refer to the nth prime number. In [And86], Andrica conjectured that

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 \tag{1}$$

holds for all n. This conjecture, and related statements, is described in [Guy04]. It is quickly checked that this holds for primes up to $4.26 \cdot 10^8$ using sagemath [Dev17].[1] It appears very likely that the conjecture is true. However it is also likely that new, novel ideas are necessary before the conjecture is decided.

Andrica's Conjecture can also be stated in terms of prime gaps. Let $g_n = p_{n+1} - p_n$ be the gap between the nth prime and the $(n+1)$st prime. Then Andrica's Conjecture is equivalent to the claim that $g_n < 2\sqrt{p_n} +$

[1]Code verifying this is available on the author's website at `http://davidlowryduda.com/?p=2430`

1. In this direction, the best known result is due to Baker, Harman, and Pintz [BHP01], who show that $g_n \ll p_n^{0.525}$.

In 1985, Sándor [Sán85] proved that

$$\liminf_{n\to\infty} \sqrt[4]{p_n}(\sqrt{p_{n+1}} - \sqrt{p_n}) = 0. \tag{2}$$

The close relation to Andrica's Conjecture (1) is clear. The first result of this note is to strengthen this result.

Theorem 1. *Let $\alpha, \beta \geq 0$, and $\alpha + \beta < 1$. Then*

$$\liminf_{n\to\infty} p_n^{\beta}(p_{n+1}^{\alpha} - p_n^{\alpha}) = 0. \tag{3}$$

We prove this theorem in Section 2. Choosing $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ verifies Sándor's result (2). But choosing $\alpha = \frac{1}{2}, \beta = \frac{1}{2} - \epsilon$ for a small $\epsilon > 0$ gives stronger results.

This theorem leads naturally to the following conjecture.

Conjecture 2. *For any $0 \leq \alpha < 1$, there exists a constant $C(\alpha)$ such that*

$$p_{n+1}^{\alpha} - p_n^{\alpha} \leq C(\alpha)$$

for all n.

A simple heuristic argument, given in Section 3, shows that this Conjecture follows from Cramér's Conjecture.

It is interesting to note that there are generalizations of Andrica's Conjecture. One can ask what the smallest γ is such that

$$p_{n+1}^{\gamma} - p_n^{\gamma} = 1$$

has a solution. This is known as the Smarandache Conjecture, and it is believed that the smallest such γ is approximately

$$\gamma \approx 0.5671481302539\ldots$$

The digits of this constant, sometimes called "the Smarandache constant," are the contents of sequence A038458 on the OEIS [Slo17]. It is possible to generalize this question as well.

Open Question. *For any fixed constant C, what is the smallest $\alpha = \alpha(C)$ such that*

$$p_{n+1}^{\alpha} - p_n^{\alpha} = C$$

has solutions? In particular, how does $\alpha(C)$ behave as a function of C?

This question does not seem to have been approached in any sort of generality, aside from the case when $C = 1$.

2. Proof of Theorem

The idea of the proof is very straightforward. We estimate (3) across prime pairs $p, p+246$, relying on the recent proof [Pol14] that infinitely many such primes exist.

Fix $\alpha, \beta \geq 0$ with $\alpha+\beta < 1$. Applying the mean value theorem of calculus on the function $x \mapsto x^\alpha$ shows that

$$\begin{aligned} p^\beta\big((p+246)^\alpha - p^\alpha\big) &= p^\beta \cdot 246\alpha q^{\alpha-1} \\ &\leq p^\beta \cdot 246\alpha p^{\alpha-1} = 246\alpha p^{\alpha+\beta-1}, \end{aligned} \tag{4}$$

for some $q \in [p, p+246]$. Passing to the inequality in the second line is done by realizing that $q^{\alpha-1}$ is a decreasing function in q. As $\alpha+\beta-1<0$, as $p \to \infty$ we see that (4) goes to zero.

Therefore

$$\liminf_{n\to\infty} p_n^\beta(p_{n+1}^\alpha - p_n^\alpha) = 0,$$

as was to be proved.

3. Further Heuristics

Cramér's Conjecture states that there exists a constant C such that for all sufficiently large n,

$$p_{n+1} - p_n < C(\log n)^2.$$

Thus for a sufficiently large prime p, the subsequent prime is at most $p + C(\log p)^2$. Performing a similar estimation as in Section 2 shows that

$$(p + C(\log p)^2)^\alpha - p^\alpha \leq C(\log p)^2 \alpha p^{\alpha-1} = C\alpha\frac{(\log p)^2}{p^{1-\alpha}}.$$

As the right hand side vanishes as $p \to \infty$, we see that it is natural to expect that Conjecture 2 is true. More generally, we should expect the following, stronger conjecture.

Conjecture 3. *For any $\alpha, \beta \geq 0$ with $\alpha+\beta<1$, there exists a constant $C(\alpha,\beta)$ such that*

$$p_n^\beta(p_{n+1}^\alpha - p_n^\alpha) \leq C(\alpha,\beta).$$

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