Solutions for Homework 1

$\boxed{1.1}$ The first several triangular-square numbers are

$$1, 36, 1225, 41616, 1413721, 48024900.$$

Finding these numbers is quite subtle and not something I expect. With a pocket calculator, you can determine $1225 = 35^2 = \underline{49 \cdot 50}$ pretty quickly, but to get further takes some good ideas.

We will return to this question in several weeks & give a complete answer. For now, it is possible (but hard) to notice that such numbers come from solutions to

$$\frac{m(m+1)}{2} = n^2$$

and that if (m,n) is a square-triangular number, then so is $(3m+4n+1, 2m+3n+1)$. ∎ [Note: I would not have noticed this myself]

$\boxed{1.2}$ Let's try a few:

$$\left.\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25 \end{aligned}\right]$$

$\vdots$

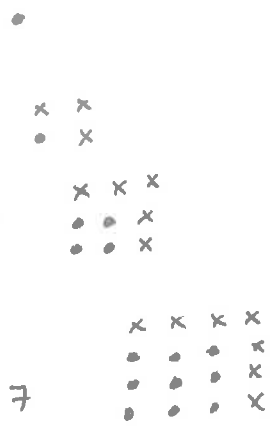
It looks like we're getting all the squares. Can we prove it?

1

1+3

1+3+5

1+3+5+7



Geometrically, it is now quite clear.

In each picture, I've added the largest odd number of the sum as "x"s, & we get squares.

It's also now clear how the general case works.

So $1 + 3 + \cdots + (2n+1) = (n+1)^2$ is our formula. ■

$\boxed{1.3}$ No, 3, 5, 7 is the only prime triplet.

The reason why is that three consecutive odd numbers are of the forms $6n+1,\ 6n+3,\ 6n+5$ or

$6n+3,\ 6n+5,\ 6n+7$ or

$6n+5,\ 6n+7,\ 6n+9$,

based on the remainder of the 1st number on division by 6.

In each possibility, one of the three is divisible by 3.

The only number ~~divisibly~~ divisible by 3 which is prime is 3 itself; so 3 must be a member of the triplet. And this leaves only 3, 5, 7.

What do you think about triplets of the form $p,\ p+2,\ p+6$?

■

1.4 This question calls for a lot of experimentation + data collection.

a As $N^2-1=(N+1)(N-1)$, these numbers are almost never prime.

b We can't factor N^2-2 like we can N^2-1. After some experimentation, it seems like N^2-1 is prime pretty often.

N	2	3	5	7	9	13
N^2-2	2	7	23	47	79	167

So it's reasonable to think it may happen infinitely often. But who knows?

c Similarly, N^2-3 doesn't factor like N^2-1. But $N^2-4=(N+2)(N-2)$, so we shouldn't expect N^2-4 to be prime very often. Some experimentation later, it seems N^2-3 is also prime pretty often.

d Generally, it seems reasonable to conjecture that if a is not a perfect square, then N^2-a is prime infinitely often. But no one knows for sure. ∎

$\boxed{2.1}$

$\boxed{a}$ If a is not a multiple of 3, then it is either $3x+1$ or $3x+2$. Similarly, if b is not a multiple of 3, then $b = 3y+1$ or $b = 3y+2$.

This gives us four possibilities.

#1 $a^2+b^2 = (3x+1)^2 + (3y+1)^2 =$

$$= 9x^2+6x+1 + 9y^2+6y+1 =$$

$$= 9x^2+6x+9y^2+6y+2.$$

Why is this bad? We need to investigate c.

If $c = 3z$, then $c^2 = 9z^2$.

If $c = 3z+1$, then $c^2 = 9z^2+6z+1$.

If $c = 3z+2$, then $c^2 = 9z^2+12z+4$.

All of these are 0 or 1 more than a multiple of 3, as

$$9z^2 = 3\cdot(3z^2)$$

$$9z^2+6z+1 = 3(3z^2+2z)+1$$

$$9z^2+12z+4 = 3(3z^2+4z+1)+1.$$

But our a^2+b^2 is 2 more than a multiple of 3.
So we can't have $a = 3x+1$, $b = 3y+1$.

#2 $a^2+b^2 = (3x+2)^2 + (3y+1)^2 = 9x^2+12x+9y^2+\overset{6y}{\cancel{12y}}+5$

#3 $a^2+b^2 = (3x+1)^2 + (3y+2)^2 = 9x^2+6x+9y^2+12y+5$

#4 $a^2+b^2 = (3x+2)^2 + (3y+2)^2 = 9x^2+12x+9y^2+12y+8$

In every case, a^2+b^2 is 2 more than a multiple of 3.
And yet c^2 cannot be (a multiple of 3) plus 2!

So we must have at least one of a or b as a multiple of 3.

$\boxed{b}$ It seems like in every primitive Pythagorean triple, exactly one of a, b, or c is a multiple of 5.

We can show this through a lot of casework. We will return to this question in a few weeks with tools which allow us to simplify the casework. ■

$\boxed{2.2}$ As ~~m divides~~ d divides m, $m=dk$ for some k.
And as d divides n, $n=dl$ for some l.

Then
$$m+n=d(k+l)$$
$$m-n=d(k-l)$$
, which are both of the form $d(*)$. So d divides $m+n$ and $m-n$. ■

$\boxed{12.7}$ $(a,b,c) \longmapsto 2c-2a$

$(3,4,5) \longmapsto 10-6 = 4$

$(5,12,13) \longmapsto 26-10 = 16$

$(7,24,25) \longmapsto 50-14 = 36$

$\vdots$

(and so on)

After experimenting, all the differences seem to be perfect squares.

To prove this, we can use what we proved:

$$\underset{\text{a ppt}}{(a,b,c)} \Longrightarrow \begin{cases} a = st \\ b = \dfrac{s^2-t^2}{2} \\ c = \dfrac{s^2+t^2}{2} \end{cases}$$

So $2c - 2a = s^2+t^2 - 2st$

$= (s-t)^2,$

which is always a perfect square. That's exactly what we wanted to prove! ■

Additional Problem (8)

As d divides a, $a = dk$ for some k. As d divides b, $b = dl$ for some l.

So $ax+by = dkx + dly = d(kx+ly)$, which is of the form $d(\text{integer})$, and so $ax+by$ is divisible by d. ■