Solutions

5.1 a, b, c, d

Pythagorean Triples have the form \((a, b, c) = (v^2 - v, 2uv, v^2 + v^2)\).

(a) If \(\gcd(u, v) = g > 1\), then \(g \mid v^2 - v^3, 2uv, v^3 + v^2\) (and in fact \(g^2 \mid v^2 - v^3, 2uv, v^3 + v^2\)), so that the triple has the common factor \(g\).

(b) \((u, v) = (3, 1)\) has \((v^2 - v^3, 2uv, v^3 + v^2) = (8, 6, 10)\), a multiple of our \((3, 4, 5)\) triangle.

(further, choosing many such \((u, v)\) are possible.)

(c) [Make a table]

(d) Some nice conditions are that

1. \(\gcd(u, v) = 1\).
2. Exactly one of \(u\) or \(v\) is even, and the other is odd.
A line through \((1,1)\) has equation
\[ y = m(x-1) + 1 = mx + (1-m), \]
where \(m\) is the slope.

The circle is given by \(x^2+y^2=2\).

To find intersections, we find simultaneous solutions
\[ \begin{cases} x^2+y^2=2 \\ y = mx + (1-m) \end{cases} \]

Let's substitute \(y = mx + (1-m)\) into the equation for the circle.
\[ x^2 + (mx + (1-m))^2 = 2, \]
which simplifies to
\[ (m^2+1)x^2 - 2(m^2-m)x + (m^2-2m-1) = 0. \]

As \(x = 1\) is a solution, we can factor out \((x-1)\) to get that
\[ (m^2+1)x^2 - 2(m^2-m)x + (m^2-2m-1) = (x-1)[(m^2+1)x - (m^2-2m-1)]. \]

So the other root is \(x = \frac{m^2-2m-1}{m^2+1}\).

The corresponding \(y\)-coordinate is
\[ y = mx + (1-m) \]
\[ = \frac{m}{m^2+1} \cdot \frac{m^2-2m-1}{m^2+1} + (1-m) \]
\[ = \frac{-m^2-2m+1}{m^2+1}. \]
So rational points on \( x^2 + y^2 = 2 \) are those points coming from rational \( m \), of the form

\[
(x, y) = \left( \frac{m^2 - 3m - 1}{m^2 + 1}, \frac{-m^2 - 3m + 1}{m^2 + 1} \right).
\]

(b) \( x^2 + y^2 = 3 \) doesn't have any rational points at all, and we need a point to start this process.

3.5

(a) We did this in class, but let's remind ourselves.

\[ n^{th} \text{ Triangular Number: } \frac{n(n+1)}{2} \]

\[ m^{th} \text{ Square Number: } m^2 \]

So we want \( m^2 = \frac{n(n+1)}{2} \), or equivalently

\[
8m^2 = 4n(n+1) = 4n^2 + 4n + 1 - 1 = (4n^2 + 4n + 1) - 1 = (2n+1)^2 - 1.
\]

Call \( x = 2n+1 \), \( y = 2m \).

Then \( 2(2m)^2 = (2n+1)^2 - 1 \) is the same as

\( 2y^2 = x^2 - 1 \), which is our hyperbola.

We want solutions where \( y \) is even and \( x \) is odd.
(b) \( x^2 - 2y^2 = 1 \)

The line through \((1, 0)\) with slope \(m\) has equation

\[ y = m(x-1). \]

Substituting into \(x^2 - 2y^2 = 1\) and solving, we find the other point

\[ (x, y) = \left( \frac{2m^2 + 1}{2m^2 - 1}, \frac{2m}{2m^2 - 1} \right). \]

(c) Writing \(m = \frac{v}{u}\), we rewrite \((x, y)\) as

\[ \left( \frac{2 \frac{v^2}{u^2} + 1}{2 \frac{v^2}{u^2} + 1}, \frac{2 \frac{v}{u}}{2 \frac{v^2}{u^2} - 1} \right) \]

which after multiplying by \(\frac{u^2}{v^2}\) becomes

\[ \left( \frac{2v^2 + u^2}{2v^2 - u^2}, \frac{2vu}{2v^2 - u^2} \right). \]

If \(v^2 - 2v^2 = 1\), the denominators are \(-1\), so that the other point (after changing signs) is

\[ (2v^2 + u^2, 2vu). \]

(d) Starting with \((3, 2)\), the next one from \((b) + (c)\) is

\[ (2 \cdot 3^2 + 3^2, 2 \cdot 3 \cdot 3) = (17, 12). \]

Starting with \((17, 12)\) gives \((577, 408)\). Then \((6657857, 470832)\).

To get square-triangular numbers from these, we need
to set $2n+1 = x$, $2m = y$. Or rather, $n = \frac{x-1}{2}$, $m = \frac{y}{2}$.

Then these values correspond to

$(3, 2) \rightarrow (\frac{3-1}{2}, \frac{2}{2}) = (1, 1)$, where $m^2 = 1$.

$(17, 12) \rightarrow (\frac{16}{2}, \frac{12}{2}) = (8, 6)$, where $m^2 = 36$.

$(577, 408) \rightarrow (289, 204)$, where $m^2 = (204)^2 = 41616$.

$(665857, 470832) \rightarrow (332928, 235416)$, where

$$m^2 = (235416)^2 = 5, 542, 069, 302, 56.$$  

{I include the 4th to show that this does get us further than we could have gotten on the 1st homework.}

(e) Starting with solution $(u, v)$, the new $y$-coordinate is $2uv$. This is always larger than $v$, so the $y$-coordinates are always increasing. Thus each time we get a new solution.  

Note: I know this was a challenging problem.

But I think it's so nice of an example of how lines and geometry can help us towards otherwise extremely challenging and impossible problems.
5.1

(a) $$\gcd(10345, 67890) = 15.$$ 

$$67890 = 5 \cdot 13578 + 12345$$
$$12345 = 2 \cdot 6165 + 15$$
$$6165 = 411 \cdot 15 + 0$$

(b) $$\gcd(54301, 9876) = 3.$$ 

$$54301 = 5 \cdot 9876 + 4941$$
$$9876 = 2 \cdot 4941 + 4935$$
$$4941 = 1 \cdot 4935 + 6$$
$$4935 = 822 \cdot 6 + 3$$
$$6 = 2 \cdot 3 + 0$$

5.4

$$\text{LCM}(8, 12) = 24$$
$$\gcd(8, 12) = 4$$
$$8 \cdot 12 = 96$$

$$\text{LCM}(20, 30) = 60$$
$$\gcd(20, 30) = 10$$
$$20 \cdot 30 = 600$$

$$\text{LCM}(51, 68) = 204$$
$$\gcd(51, 68) = 17$$
$$51 \cdot 68 = 3444$$

$$\text{LCM}(23, 18) = 414$$
$$\gcd(23, 18) = 1$$
$$23 \cdot 18 = 414$$

The relationship is $$\gcd(a, b) \cdot \text{LCM}(a, b) = a \cdot b.$$ Did you know this already? ☝️

5.5

(a) 21 → 64 → 32 → 16 → 8 → 4 → 2 → 1  length 8

13 → 40 → 20 → 10 → 5 → 16 → 8 → 4 → 2 → 1  length 10

For 31 ... this was a bit cruel. The length is 107.

Isn't it surprising how large these numbers can get?

The next number with longer length is 41, of length 110. ☝️

(b) What do you think? Many think it always terminates in a $$4 \rightarrow 2 \rightarrow 1$$ loop. We don't know, though.

(c) Notice 8k+4 → 4k+2 → 2k+1 → 6k+4, so it takes 3 steps before they fall into the same sequence. ☝️
The only certainty is that the steps must look something like

\[ \_ = \_ \cdot A + q \]
\[ A = \_ \cdot q + 4 \]
\[ q = 2 \cdot 4 + 1 \]
\[ 4 = 1 \cdot 4 + 0 \]

where the blanks are open, and the "A" spots are the same (and A > 9). One possibility is

\[ 58 = 1 \cdot 49 + 9 \]
\[ 49 = 5 \cdot 9 + 4 \]
\[ 9 = 2 \cdot 4 + 1 \]
\[ 4 = 4 \cdot 1 + 0 \]

Another is

\[ 197 = 2 \cdot 94 + 9 \]
\[ 94 = 10 \cdot 9 + 4 \]
\[ 9 = 2 \cdot 4 + 1 \]
\[ 4 = 4 \cdot 1 + 0 \]

but there are infinitely many.