# ON A FUNCTIONAL EQUATION FOR $\sum_{n\geq 1} \frac{a(n)}{n^s} e\left(\frac{n\overline{r}}{c}\right)$

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In this note, I remind myself of the functional equations for the $L$-functions $\sum_{n\geq 0} \frac{a(n)}{n^s}$ and $\sum_{n\geq 0} \frac{a(n)}{n^s} e(\frac{n\overline{r}}{c})$, where $\overline{r}$ is the multiplicative inverse of $r$ mod $c$.

Let $f$ be a weight $k$ modular cusp form, so that $f$ satisfies $f(\gamma z) = (cz+d)^k f(z)$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (and $\gamma$ will always be this), and where the Fourier expansion of $f$ is $f(z) = \sum_{n\geq 1} a(n) e^{2\pi i n z} = \sum_{n \geq 1} a(n) e(nz)$. Firstly, we note the relationship between the first L-function $\sum \frac{a(n)}{n^s}$ and the integral $\int_0^\infty f(iy) y^s \frac{\mathrm{d}y}{y}$.

$$\begin{aligned}
\int_0^\infty f(iy) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} &= \int_0^\infty \sum_{n\geq 1} a(n) e(iny) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} \\
&= \sum_{n\geq 1} \int_0^\infty a(n) e^{-2\pi n y} y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} && \left(y \mapsto \frac{y}{2\pi n}\right) \\
&= \frac{1}{(2\pi n)^{s+\frac{k-1}{2}}} \sum_{n\geq 1} \int_0^\infty a(n) e^{-y} y^{s+\frac{k+1}{2}-1} \frac{\mathrm{d}y}{y} \\
&= \sum_{n\geq 1} \frac{a(n)}{(2\pi n)^{s+\frac{k-1}{2}}} \Gamma(s + \frac{k+1}{2}) \\
&= \frac{\Gamma(s+\frac{k+1}{2})}{(2\pi)^{s+\frac{k-1}{2}}} \sum_{n\geq 1} \frac{a(n)}{n^{s+\frac{k-1}{2}}} := \Lambda(f,s),
\end{aligned}$$

where we might notice that the notation has the $\frac{k-1}{2}$ built into the $\Lambda$ expression.

As $f$ is modular, $f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) iy\right) = (iy)^k f(iy) = f(\frac{-1}{iy}) = f(\frac{i}{y})$, so we also have that

$$\begin{aligned}
\int_0^\infty f(iy) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} &= \int_0^\infty f\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) iy\right) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} \\
&= i^{-k} \int_0^\infty f\left(\frac{i}{y}\right) y^{s+\frac{k-1}{2}-k} \frac{\mathrm{d}y}{y} \\
&= i^{-k} \int_0^\infty f\left(\frac{i}{y}\right) y^{s+\frac{-k-1}{2}} \frac{\mathrm{d}y}{y} && \left(y \mapsto \frac{1}{y}\right) \\
&= i^{-k} \int_0^\infty f(iy) y^{-s+\frac{k+1}{2}} \frac{\mathrm{d}y}{y} \\
&= i^{-k} \sum_{n\geq 1} \int_0^\infty a(n) e^{-2\pi n y} y^{\frac{k+1}{2}-s} \frac{\mathrm{d}y}{y} && \left(y \mapsto \frac{y}{2\pi n}\right) \\
&= \sum_{n\geq 1} \frac{i^{-k} a(n)}{(2\pi n)^{\frac{k+1}{2}-s}} \int_0^\infty e^{-y} y^{(1-s)+\frac{k+1}{2}-1} \frac{\mathrm{d}y}{y} \\
&= \frac{i^{-k}\Gamma(1-s+\frac{k+1}{2})}{(2\pi)^{(1-s)+\frac{k-1}{2}}} \sum_{n\geq 1} \frac{a(n)}{n^{(1-s)+\frac{k-1}{2}}} = i^{-k} \Lambda(f, 1-s),
\end{aligned}$$

giving us the following proposition.

**Proposition 1.** $\Lambda(f,s) = i^{-k}\Lambda(f,1-s)$, *where*

$$\Lambda(f,s) = \frac{\Gamma(s+\frac{k+1}{2})}{(2\pi)^{s+\frac{k-1}{2}}} \sum_{n\geq 1} \frac{a(n)}{n^{s+\frac{k-1}{2}}}. \tag{1}$$

Now we do the same sort of idea for the twisted function cusp form $\sum_{n\geq 1} a(n)e\left(\frac{n\overline{r}}{c}\right)e(nz)$. To examine this, we will look at the integral $\int_0^\infty f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right) y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y}$.

$$\begin{aligned}
\int_0^\infty f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right) y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y} &= \sum_{n\geq 1}\int_0^\infty a(n)e\left(\frac{n\overline{r}}{c}+\frac{iny}{c}\right) y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y} && \left(y\mapsto \frac{cy}{2\pi n}\right)\\
&= \sum_{n\geq 1}\left(\frac{c}{2\pi n}\right)^{s+\frac{k-1}{2}}\int_0^\infty a(n)e^{\frac{2\pi i n\overline{r}}{c}}e^{-y}y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y}\\
&= \left(\frac{c}{2\pi}\right)^{s+\frac{k-1}{2}}\sum_{n\geq 1}\frac{a(n)}{n^{s+\frac{k-1}{2}}}e\left(\frac{n\overline{r}}{c}\right)\int_0^\infty e^{-y}y^{s+\frac{k+1}{2}-1}\frac{\mathrm{d}y}{y}\\
&= \left(\frac{c}{2\pi}\right)^{s+\frac{k-1}{2}}\Gamma\left(s+\frac{k+1}{2}\right)\sum_{n\geq 1}\frac{a(n)}{n^{s+\frac{k-1}{2}}}e\left(\frac{n\overline{r}}{c}\right)\\
&:= \Lambda\left(f,s,\frac{\overline{r}}{c}\right),
\end{aligned}$$

and where we see that my notation suggests that

$$L\left(f,s,\frac{\overline{r}}{c}\right) = \sum_{n\geq 1}\frac{a(n)}{n^{s+\frac{k-1}{2}}}e\left(\frac{n\overline{r}}{c}\right).$$

Now, as $f$ is modular, it is invariant under the action of $\left(\begin{smallmatrix} r & \alpha \\ c & \overline{r}\end{smallmatrix}\right)$. So on the one hand,

$$f\left(\begin{pmatrix} r & \alpha\\ -c & \overline{r}\end{pmatrix}\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right)\right) = f\left(-\frac{r}{c}+\frac{1}{cy}i\right),$$

while on the other hand

$$f\left(\begin{pmatrix} r & \alpha\\ -c & \overline{r}\end{pmatrix}\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right)\right) = \left(-c\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right)+\overline{r}\right)^k f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right) = (-iy)^k f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right).$$

So, now that we know that $(-iy)^{-k}f\left(-\frac{r}{c}+\frac{1}{cy}i\right) = f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right)$, we can say that

$$\begin{aligned}
\int_0^\infty f\left(\frac{\overline{r}}{c}+\frac{iy}{c}\right) y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y} &= \int_0^\infty (-iy)^{-k}f\left(-\frac{r}{c}+\frac{1}{cy}i\right)y^{s+\frac{k-1}{2}}\frac{\mathrm{d}y}{y}\\
&= (-i)^{-k}\int_0^\infty f\left(-\frac{r}{c}+\frac{1}{cy}i\right)y^{s+\frac{-k-1}{2}}\frac{\mathrm{d}y}{y}\\
&= (-i)^{-k}\sum_{n\geq 0}a(n)e\left(\frac{-rn}{c}\right)\int_0^\infty e\left(\frac{in}{cy}\right)y^{s+\frac{-k-1}{2}}\frac{\mathrm{d}y}{y} && \left(y\mapsto\frac{2\pi n}{cy}\right)\\
&= (-i)^{-k}\left(\frac{c}{2\pi}\right)^{\frac{k+1}{2}-s}\sum_{n\geq 1}\frac{a(n)}{n^{\frac{k+1}{2}-s}}e\left(\frac{-rn}{c}\right)\int_0^\infty e^{-y}y^{\frac{k+1}{2}-s}\frac{\mathrm{d}y}{y}\\
&= (-i)^{-k}\left(\frac{c}{2\pi}\right)^{(1-s)+\frac{k-1}{2}}\Gamma\left((1-s)+\frac{k+1}{2}\right)\sum_{n\geq 1}\frac{a(n)}{n^{(1-s)+\frac{k-1}{2}}}e\left(\frac{-rn}{c}\right)\\
&= (-i)^{-k}\Lambda\left(f,1-s,\frac{-r}{c}\right).
\end{aligned}$$

**Proposition 2.**

$$\Lambda\left(f, s, \frac{\overline{r}}{c}\right) = (-i)^{-k}\Lambda\left(f, 1-s, \frac{-r}{c}\right), \tag{2}$$

*where*

$$\left(\frac{c}{2\pi}\right)^{s+\frac{k-1}{2}}\Gamma\left(s+\frac{k+1}{2}\right)\sum_{n\geq 1}\frac{a(n)}{n^{s+\frac{k-1}{2}}}e\left(\frac{n\overline{r}}{c}\right) := \Lambda\left(f, s, \frac{\overline{r}}{c}\right)$$

So what, we might ask? This will allow us to convert the $\overline{r}$ part of an annoying Kloosterman sum into a much nicer $r$, giving us something that we can handle, in a forthcoming application. (And is something that we've done enough that I want to have the computation written down completely and nicely).