Response to bnelo12

answered by mixedmath
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bnelo12 writes (slightly paraphrased)

> Can you explain exactly how $1+2+3+4+\ldots = -\frac{1}{12}$ in the context of the Riemann ζ function?

We are going to approach this problem through a related problem that is easier to understand at first. Many are familiar with summing geometric series

$$g(r) = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r},$$

which makes sense as long as $|r| < 1$. But if you're not, let's see how we do that. Let $S(n)$ denote the sum of the terms up to r^n, so that

$$S(n) = 1 + r + r^2 + \ldots + r^n.$$

Then for a finite n, $S(n)$ makes complete sense. It's just a sum of a few numbers. What if we multiply $S(n)$ by r? Then we get

$$rS(n) = r + r^2 + \ldots + r^n + r^{n+1}.$$

Notice how similar this is to $S(n)$. It's very similar, but missing the first term and containing an extra last term. If we subtract them, we get

$$S(n) - rS(n) = 1 - r^{n+1},$$

which is a very simple expression. But we can factor out the $S(n)$ on the left and solve for it. In total, we get

$$S(n) = \frac{1-r^{n+1}}{1-r}. \tag{1}$$

This works for any natural number n. What if we let n get arbitrarily large? Then if $|r| < 1$, then $|r|^{n+1} \to 0$, and so we get that the sum of the geometric series is

$$g(r) = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}.$$

But this looks like it makes sense for almost any r, in that we can plug in any value for r that we want on the right and get a number, unless $r = 1$. In this sense, we might say that $\frac{1}{1-r}$ **extends** the geometric series $g(r)$, in that whenever $|r| < 1$, the geometric series $g(r)$ agrees with this function. But this function makes sense **in a larger domain** then $g(r)$.

People find it convenient to abuse notation slightly and call the new function $\frac{1}{1-r} = g(r)$, (i.e. use the same notation for the extension) because any time you might want to plug in r when $|r| < 1$, you still get the same value. But really, it's not true that $\frac{1}{1-r} = g(r)$, since the domain on the left is bigger than the domain on the right. This can be confusing. It's things like this that cause people to say that

$$1 + 2 + 4 + 8 + 16 + \ldots = \frac{1}{1-2} = -1,$$

simply because $g(2) = -1$. This is conflating two different ideas together. What this means is that the function that extends the geometric series takes the value -1 when $r = 2$. But this has nothing to do with actually summing up the 2 powers at all.

So it is with the ζ function. Even though the ζ function only makes sense at first when $\mathrm{Re}(s) > 1$, people have extended it for almost all s in the complex plane. It just so happens that the great functional equation for the Riemann ζ function that relates the right and left half planes (across the line $\mathrm{Re}(s) = \frac{1}{2}$) is

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{2}$$

where Γ is the gamma function, a sort of generalization of the factorial function. If we solve for $\zeta(1-s)$, then we get

$$\zeta(1-s) = \frac{\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)}.$$

If we stick in $s = 2$, we get

$$\zeta(-1) = \frac{\pi^{-1}\Gamma(1)\zeta(2)}{\pi^{\frac{-1}{2}}\Gamma\left(\frac{-1}{2}\right)}.$$

We happen to know that $\zeta(2) = \frac{\pi^2}{6}$ (this is called Basel's problem) and that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We also happen to know that in general, $\Gamma(t+1) = t\Gamma(t)$ (it is partially in this sense that the Γ function generalizes the factorial function), so that $\Gamma(\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{-1}{2})$, or rather that $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi}$. Finally, $\Gamma(1) = 1$ (on integers, it agrees with the one-lower factorial).

Putting these together, we get that

$$\zeta(-1) = \frac{\pi^2/6}{-2\pi^2} = \frac{-1}{12},$$

which is what we wanted to show. $\diamondsuit$

The information I quoted about the Gamma function and the zeta function's functional equation can be found on Wikipedia or any introductory book on analytic number theory. Evaluating $\zeta(2)$ is a classic problem that has been in many ways, but is most often taught in a first course on complex analysis or as a clever iterated integral problem (you can prove it with Fubini's theorem). Evaluating $\Gamma(\frac{1}{2})$ is rarely done and is sort of a trick, usually done with Fourier analysis.

This writing (and several others) can be found on my blog at davidlowryduda.com.