

1) (a) $\sum \frac{n}{\sqrt{2n^5 + 6n^2}} \sim \sum \frac{n}{n^{5/2}} = \sum \frac{1}{n^{3/2}}$ converges by p-series

limit comparison

$$n \cdot \frac{n^{5/2}}{\sqrt{2n^5 + 6n^2}} \rightarrow \frac{1}{\sqrt{2}}$$

(b) Similarly, (b) converges by limit comparison to $\sum \frac{1}{n^{3/2}} = \sum \frac{1}{\sqrt{n}}$.

(c) $\sum_{n=1}^{\infty} \frac{n^c}{\sqrt{2n^5 + 6n^2}} \sim \sum \frac{n^c}{n^{5/2}}$

limit comparison, as $\frac{n^c \cdot n^{5/2}}{\sqrt{2n^5 + 6n^2}} \rightarrow \frac{1}{\sqrt{2}}$ as above.

+ $\sum \frac{n^c}{n^{5/2}} = \sum \frac{1}{n^{5/2-c}}$ which is a p-series, + so converges iff $5/2 - c > 1$, i.e. $5/2 - 1 > c$, i.e.

$$\boxed{c < 3/2}$$

2) We'll skip to (c).

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^c}$$

integral test:

$$\int_2^{\infty} \frac{1}{x(\ln x)^c} dx = \int_{\ln 2}^{\infty} \frac{1}{u^c} du = \frac{u^{-c+1}}{-c+1} \Big|_{\ln 2}^{\infty}$$

$u = \ln x$
 $du = \frac{1}{x}$

which converges iff $-c+1 < 0$, i.e. $\boxed{c > 1}$.

3) $2x^4 \leq f(x) \leq 4x^4$

$$\sum \frac{2n^4}{n^c} \leq \sum \frac{f(n)}{n^c} \leq \sum \frac{4n^4}{n^c} \implies \sum \frac{2}{n^{c-4}} < \sum \frac{f(n)}{n^c} < \sum \frac{4}{n^{c-4}}$$

diverges if $c \leq 4$ converges if $c > 4$

so it converges iff $c > 4$ by 2 basic comparisons.

4) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n^c} \sim \sum_{n=1}^{\infty} \frac{1}{n^c}$ limit comparison, as

L'Hôpital's Rule

$$\lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \cdot \frac{n^c}{n^c} = \lim_{n \rightarrow \infty} \frac{\cos(\frac{1}{n}) \cdot \frac{1}{n^2}}{\frac{1}{n^2}}$$

+ $\lim_{n \rightarrow \infty} \cos(\frac{1}{n}) = \cos(0) = 1.$

So $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n^c} \sim \sum_{n=1}^{\infty} \frac{1}{n^{c+1}}$ which converges by p-series iff $c > 0$.