Response to fattybake
answered by mixedmath
shameless plug: mixedmath.wordpress.com, davidlowryduda.com

We want to understand the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(1+t^2)^n}. \tag{1}$$

Although fattybake mentions the residue theorem, we won't use that at all. Instead, we will be very clever.

We will do a technique that was once very common (up until the 1910s or so), but is much less common now: let's multiply by $\Gamma(n) = \int_0^\infty u^n e^{-u} \frac{\mathrm{d}u}{u}$. This yields

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{u}{1+t^2}\right)^n e^{-u} \mathrm{d}t \frac{\mathrm{d}u}{u} = \int_{-\infty}^\infty \int_0^\infty \left(\frac{u}{1+t^2}\right)^n e^{-u} \frac{\mathrm{d}u}{u} \mathrm{d}t, \tag{2}$$

where I interchanged the order of integration because everything converges really really nicely. Do a change of variables, sending $u \mapsto u(1+t^2)$. Notice that my nicely behaving measure $\mathrm{d}u/u$ completely ignores this change of variables, which is why I write my Γ function that way. Also be pleased that we are squaring t, so that this is positive and doesn't mess with where we are integrating. This leads us to

$$\int_{-\infty}^\infty \int_0^\infty u^n e^{-u+-ut^2} \frac{\mathrm{d}u}{u} \mathrm{d}t = \int_0^\infty \int_{-\infty}^\infty u^n e^{-u+-ut^2} \mathrm{d}t \frac{\mathrm{d}u}{u}, \tag{3}$$

where I change the order of integration again. Now we have an inner t integral that we can do, as it's just the standard Gaussian integral (google this if this doesn't make sense to you). The inner integral is

$$\int_{-\infty}^\infty e^{-ut^2} \mathrm{d}t = \sqrt{\pi/u}.$$

Putting this into the above yields

$$\sqrt{\pi} \int_0^\infty u^{n-1/2} e^{-u} \frac{\mathrm{d}u}{u}, \tag{4}$$

which is exactly the definition for $\Gamma(n - \frac{1}{2}) \cdot \sqrt{\pi}$.

But remember, we multiplied everything by $\Gamma(n)$ to start with. So we divide by that to get the result:

$$\int_{-\infty}^\infty \frac{\mathrm{d}t}{(1+t^2)^n} = \frac{\sqrt{\pi}\Gamma(n - \frac{1}{2})}{\Gamma(n)} \tag{5}$$