REVIEW EXERCISES

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Contents

1. Introduction

We have now covered three-fifths of the material from this course. While in many ways, this course has been cumulative and we have revisited much of the earlier material repeatedly, there are things that have been left out. In this packet, we will review some of the problem-types that we have come across and methods of finding their solution.

We will also take this time to combine our new skills, such as our knowledge of trigonometric identities and transcendental functions, with some of our old skills.

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2. Elementary Functions

We started by reviewing basic factoring and graphing linear equations. We then worked on developing the algebraic and graphical qualities of polynomials. We learned to solve any quadratic equation that we will ever see (any cubic too, though that's much harder). Later came the Factor Theorem and the Rational Root Theorem, allowing us to solve more. This all culminated with rational functions. This first set of exercises covers this material.

2.1. **Factoring and Solving Quadratics.**

Example 2.1. One tool that we use with high frequency is factoring. We cannot get away from factoring, it turns out. Many tools revolve around factoring. We'll look at three major factoring tools here.

(1) $(x+y)^2 = x^2 + 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$
(2) $(x-y)(x+y) = x^2 - y^2$
(3) Completing the square

Example 2.2. We can use $(x+y)^2 = x^2 = 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$ both ways. For example, when we see an expression that we'd like to expand, like $(\sin x + \cos x)^2$, we can immediately say that it is $\sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + 2\sin x\cos x$ without any FOILing. (further, if we are exceptionally clever, we might remember that $2\sin x\cos x = \sin(2x)$).

On the other had, we can factor functions quickly too. For example, when we are asked to find the roots of $e^{2x} - 4e^x + 4$, we recognize this as $(e^x - 2)^2$. Thus there is a double root at $x = \ln 2$.

Exercise 2.3. Expand the following without directly multiplying it out:

- $(3x + 2y)^2$
- $(5x - 4)^2$
- $(x + \cos x)^2$
- $(\sec x - \sin x)^2$
- $(e^x + e^{-x})^2$

Exercise 2.4. Factor the following:

- $\cos^2 x + 2\cos x \tan x + \tan^x x$
- $2e^{2x} + 6\sqrt{2}e^x + 9$
- $\sin^2 x + 8\sin x + 16$
- $2 + x^2 + x^{-2}$

Example 2.5. It is usually very easy to see cases where we have $(x-y)(x+y)$ and rewrite it as $x^2 - y^2$, but we sometimes need to approach it in the opposite direction. For example, if we want to find the roots of $\sin^2 x - 1/2$, we can do this quickly and easily with this factoring method. $\sin^2 x - 1/2 = (\sin x - 1/\sqrt{2})(\sin x + 1/\sqrt{2})$, and thus the solutions are $x = \pi/4 + n\pi/2$.

Exercise 2.6. Factor the following:

- $3x^2 - 2y^2$ (just because everything starts as 'squares' doesn't mean that they're the squares of pretty numbers)
- $4e^{2x} - 9$
- $2\tan x - 4$

Further, if we want to factor over the complex numbers, we might notice that $x^2 + y^2 = (x + iy)(x - iy)$. So we cam factor the following:

- $4x^2 + 9y^2$
- $2x^2 + 16y^2$

Example 2.7. There is a general form for completing the square that always works. If we have $x^2 + ax + b$, we can note that $x^2 + ax + \frac{a^2}{4} - \frac{a^2}{4} + b = (x+\frac{a}{2})^2 - \frac{a^2}{4} + b$. For example, $x^2+3x+5 = x^2+3x+\frac{9}{4}-\frac{9}{4}+5 = (x+\frac{3}{2})^2-\frac{9}{4}+5$. If we believe this pattern, we could skip the middle.

For example, if we had $x^2+10x+3$, we could write this as $(x+\frac{10}{2})^2-25+3$ without any of the middle expansions.

Exercise 2.8. Complete the square on the following:

- $x^2 + 18x + 2$
- $x^2 + 5x + 3$
- $\sin^2 x + 3\sin x + 2$
- $2x^2 + 3x + 5$ (the task here is to remember how to deal with the leading 2)

The goal is for these factoring techniques to feel like second nature. The less one needs to think about them, the better. The task of finding roots is largely equivalent to factoring, due to the Factor Theorem. In short, this says that if $p(x)$ is a polynomial and $p(r) = 0$, then $p(x) = (x-r)q(x)$ where $q(x)$ is a smaller degree polynomial. Thus we can pull out a linear factor of $(x - r)$.

Example 2.9. Anytime we see a quadratic, we should be happy. We can solve quadratics, always. Through factoring or the quadratic formula, we can always solve quadratics. In fact, we can solve them quickly. One shouldn't need to spend more than a minute on a quadratic in the form $ax^2+bx+c = 0$.

Exercise 2.10. Solve the following quadratics:

- $x^2 + 5x + 18 = 0$
- $4x^2 + 3x + 2 = 0$
- $2\sin^2 x + 3\sin x + 1 = 0$
- $3\cos^2 x + 8\sin x + 1 = 0$
- $2e^{4x} - 13e^{2x} + 1 = 0$

In cases (like above), remember that $\cos^2 x + \sin^2 x = 1$, and this can be modified to relate $\csc^2 x$ to $\cot^2 x$, or to relate $\sec^2 x$ to $\tan^2 x$. It's important to recognize 'hidden quadratics,' and to do the necessary work to transform a quadratic into a form that you can solve.

Exercise 2.11. Solve the following quadratics:

- $5\cot^2 x + 14\csc x + 1 = 0$
- $2\cos^2 x + 4\sin x + 2 = 0$
- $\tan^2 x + 5\sec x + 3 = 0$
- $(x-2)^2 + 2x + 4 = (x-1)$
- $(x-3)^3 + (x-1)(x+1) = x^3 + 1$
- $\sqrt{x-2} + \sqrt{x+2} = 2$
- $\sqrt{x+1} + \sqrt{x-4} = \sqrt{x} + 5$

2.2. **Polynomial Inequalities.** Perhaps the best way of solving polynomial inequalities is to find its roots, make a sign chart, and just test on each side of each root. It is almost certainly the fastest. This is our general method:

Example 2.12. When we see a polynomial inequality $p(x) \geq 0$, we find the roots $r_1, \ldots, r_n$ of the polynomial. We then draw a number line, and see if the polynomial is positive or negative between each pair of roots. As polynomials are continuous, they will only change signs at roots. We use this to decide on our inequality. As an aside, I want to mention that this is one of the easiest types of questions to merit partial credit as long as you show your work, if you're in a graded situation.

Example 2.13. Let us solve the quadratic inequality $3x^2 + 4x \geq 1$. First, we gather ever everything to one side. So we want to solve $3x^2 + 4x - 1 \geq 0$. Where are the roots of the quadratic $3x^2 + 4x - 1$? This doesn't immediately seem to factor nicely, so we use the quadratic formula: the roots are $x_+, x_- = \dfrac{-4 \pm \sqrt{16+12}}{6}$. These roots split the real line into the three regions $(-\infty, x_-), (x_-, x_+), (x_+, \infty)$. We need to check the sign of our polynomial on each of the three regions. Using, for example, $-100, 0, 100$, we see that the signs go $+ - +$. Thus the inequality's solution is x in $\left(-\infty, \dfrac{-4-\sqrt{28}}{6}\right]$ and $\left[\dfrac{-4+\sqrt{28}}{6}, \infty\right)$.

Exercise 2.14. Solve the following quadratic inequalities.

- $8x^2 - x > 3$
- $9x^2 + 6x > 1$
- $4x^2 \geq 4$

The same idea works for higher degree polynomials as well. The task is the same: find the roots, make a number line, identify regions where the polynomial is positive and negative, and use this to find your answer. Also remember - **it is not always the case that the sign switches positive negative positive negative.**

Exercise 2.15. Solve the following polynomial inequalities. These can be done with factoring.

- $x^2 + 4x - 6 \geq 6$
- $2x^2 + x - 15 < 0$

- $-x^2 + 2x + 3 \geq 0$
- $x^3 - x^2 - 16x + 16 \leq 0$
- $x^3 - x^2 - 16x + 16 \geq 36$
- $x^4 - x^2 - 20 > 0$

Exercise 2.16. Solve the following polynomial inequalities. You may have to use other tools, such as the rational root theorem or factor theorem, to proceed here.

- $x^4 - x^3 - 2x - 4 > 0$
- $x^5 - x^4 - 3x^3 + 5x^2 - 2x$

2.3. **Rational Functions.** In many ways, understanding rational functions comes down to understanding polynomials. Once we understand polynomials and, in particular, identifying where they are positive, negative, or zero, we know a tremendous amount about rational functions.

The general method of attacking rational functions is to find the zeroes of the numerator and denominator, set up a sign chart with these zeroes as the important places, and to identify where the rational function will be positive and where it will be negative. Zeroes of the numerator lead to zeroes of the rational function. Zeroes of the denominator lead to vertical asymptotes of the rational function. If there is the same zero in the numerator and denominator, then there might be a hole.

The only bit remaining with respect to rational functions is to understand their limiting behavior. This falls into a few different categories: there might be a horizontal asymptote, a slant asymptote, or no asymptote.

Example 2.17. Consider the rational function $f(x) = \dfrac{x^2 + 3x + 5}{x^3 + x + 1}$, and suppose we want to find its limiting behavior. If we think of really large x, then x^3 is much larger than x^2. In general, if the degree of the denominator is greater than the degree of the numerator, than the limiting behavior is a horizontal asymptote at $y = 0$. That is the case here.

Example 2.18. Consider the rational function $g(x) = \dfrac{x^3 + 3x + 1}{4x^3 + 1}$, and suppose we want to find its limiting behavior. If we think of really large x this time, we can't use the same trick as above. Now the degree of the numerator and denominator are the same. But for really large x, everything except the x^3 and $4x^3$ terms matter less and less. In general, if the degrees of the denominator and numerator are the same, then there is a horizontal asymptote. For $g(x)$, we expect $g(x) \approx \frac{1}{4}$ for really large x, as the x^3 term of the numerator gets divided by $4x^3$ in the denominator. This leads to the general fact that the horizontal asymptote in these cases will be at $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.

Example 2.19. Consider the rational function $h(x) = \dfrac{x^3+3x+1}{x^2+1}$. The degree of the numerator is exactly one more than the degree of the denominator. Using polynomial long division, we see that $h(x) = x + \dfrac{2x+1}{x^2+1}$, so that for large x the polynomial behaves just like x (the $\dfrac{2x+1}{x^2+1} \to 0$ as x gets big). We call the line x in this case the *slant asymptote*, and we find it in general by performing polynomial long division.

Example 2.20. Consider the rational function $j(x) = \dfrac{x^5+3x^2+1}{x^2+1}$. Polynomial long division would reveal limiting behavior similar to a cubic, as the degree of the numerator is 5 and the degree of the denominator is only 2. We don't care about 'curved' asymptotes in this course, so all that we care about here is whether the function goes to ∞ or $-\infty$ as $x \to \infty$ and $x \to -\infty$.

Exercise 2.21. Find the zeroes, vertical asymptotes, holes, horizontal asymptotes, and slant asymptotes of the following rational functions. Sketch the results.

(1) $\dfrac{x^2-5x+4}{x^2-4}$
(2) $\dfrac{2x^2-5x+2}{4x^2-2x-12}$
(3) $\dfrac{2x^3-x^2-2x+1}{x^2+3x+2}$
(4) $\dfrac{2x^3+x^2-8x-4}{x^2-4x+2}$ (similar to, but not the same as, the previous)

Exercise 2.22. Let's see a sort of way in which rational functions might come up. Certain professions, such as any sort of manufacturing or chemical engineer, need to worry about particular types of problems that we call "mixing problems." Suppose, for instance, that a large tank contains 50 liters of a 75%/25% water/sodium benzoate solution. We want a larger concentration of sodium benzoate, but it's challenging and expensive to get pure sodium benzoate. But it's easy to get a 75%/25% sodium benzoate/water mixture. So we pour x liters of this new mixture into the tank.

- Show that the new concentration C (starting at 0.25 and changing because we are adding liquid with a 0.75 concentration) is given by $C = \dfrac{3x+50}{4(x+50)}$
- Find the limiting behavior of this system (for positive x only - the implied domain of this model is for x positive only. Why is that?).
- Does this limiting behavior make sense?

In fact, mixing problems are very important. But to be fair, this would be one of the easiest mixing problems out there. Chemical engineers, for example, have to work with different concentrations of different materials interacting with each other - and different concentrations change the rate of chemical reaction and interaction as well. There is some intense math there - but this is where it starts.

Exercise 2.23. In my work, I happen to use rational functions quite a bit. There are some miraculous properties of rational functions. For better or worse, we look at two of them here.

(1) Often, math asks meta-type questions: instead of "what is the solution?" it might ask "when is this solveable?" For example, for what k is the equation

$$x^2 + (1-3k)x + (2-k) = 0 \tag{2.1}$$

solveable for real-valued x? *To do this, 'solve' for k. You'll get a rational function. Find the range of that rational function, and this will be the exact vales of k for which equation (2.1) is solveable.*

(2) This introduces a surprising relationship between rational functions, matrices, and complex numbers. Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can associate a rational function $f(z) = \dfrac{az+b}{cz+d}$ on the complex numbers. There are some stunning things here: if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then the rational function $f(z) = \dfrac{az+b}{cz+d}$ is invertible with inverse $f^{-1}(z) = \dfrac{ez+f}{gz+h}$. This is not at all obvious, and is a bit surprising. If you recall the geometry of complex numbers, and remember that multiplying has to do with a certain rotation and scaling operation, then one can view the associated rational functions to matrices as doing a certain rotation, scaling, shifting, and then doing another roation, scaling, and shifting. The work I do uses this sort of interplay extensively, and this hints at two pervasive concepts of higher mathematics: we find connections between different objects, ultimately learning more about everything involved; and we let things 'act' on other things (in this case, matrices are 'acting' on the complex plane) and through these actions, we learn more about both what's being acted upon and the actor.

2.4. **Exponentials and Logs.**

Exercise 2.24. Review the basic definitions and properties of exponentials and logarithms. Also review the change-of-base formula.

Example 2.25. Our key interest with exponentials was with modeling certain types of growth. The easiest to remember is compounding interest. If an initial payment of P is put into an account that grows as r percent interest that compounds n times a year, then after t years, there will be $P(1 + \frac{r}{n})^{nt}$ in the account. If the interest compounds continuously, there will be Pe^{rt} in the account.

Exercise 2.26. Find the amount of money in the accounts at the designated time. It might be good if you try to estimate the results, to compare with your intuition.

(1) Initial amount: 300, interest rate: 10%, compounded quarterly, time: 5 years
(2) Initial amount: 1000, interest rate: 5%, compounded monthly, time: 7 years
(3) Initial amount: 4000, interest rate: 4%, compounded bi-annually (twice a year), time: 20 years

Exercise 2.27. Solve the following exponential equations:

(1) $2P_0 = P_0e^{.07x}$ (this is asking how long a continuously-compounded interest account would take to double with 7% interest)
(2) $4P_0 = P_0e^{.07x}$ (does the answer to this make sense, considering the above?)
(3) $xe^x + e^x = 0$
(4) $x \ln x + x = 0$
(5) $\ln x + \sqrt{x} - 1 = 0$ *(This is much harder than the others. Hint: see if you can find a solution, and then justify that it's the only solution. Bigger hint: show that this function is always increasing, and thus is one-to-one)*

Example 2.28. This example serves as a reminder of the various exponential models that we have come across. In particular, we looked at five different type of model.

(1) Exponential growth: a function of the form $f(x) = ae^{bx}$ with $b > 0$. It's good to remember that exponential growth is larger than any polynomial growth.
(2) Exponential decay: a function of the form $f(x) = ae^{-bx}$ with $b < 0$.
(3) Logistic growth: a function of the form $f(x) = \dfrac{a}{1 + be^{rx}}$, $r \neq 0$. It's good to remember that this describes population growth with an upper and lower limit. The numerator, a, is called the 'carrying capacity' of the system.
(4) Logarithmic growth: a function of teh form $f(x) = a + b \ln x$. It's good to remember that logarithmic growth is slower than any polynomial growth.

Exercise 2.29. Let's do an experiment. Suppose you are in college debt, a situation which forces some to get a new loan every 6 months for 4 to 5 years. After some amount of time, you might have to pay back 9 or so different loans, each with their own interest rates. Think to yourself about the following: what's the best way to pay it back? Choose the largest interest account and pay that one off? Distribute money across several accounts? Pay off the interest on each, but focus on one or another? *This exercise will be a bit computation heavy, so I recommend that you pull out your calculator, some paper, and keep great notes and a table. It is these notes/table that I'll want to see*

(1) Let us suppose each of the 9 loans is for 4000 dollars, and they have the following anual interest rates: $3\%, 3.5\%, 4\%, 4.4\%, 4.8\%, 5\%, 5.2\%, 5.2\%, 5.4\%$, compounding continuosly (it will give a good approximation). Let us also suppose that we have 500 dollars available per month to pay into these loans, and we invest these at the end of the month, each month.

(2) First, let's see what happens if we use a dumb payback scheme: pay off the smallest interest first, and then progress higher. The smallest loan debt would grow like $4000e^{.03t}$. After one month, the debt on this account would be $4000e^{.03(1/12)}$. This is about \$4010.01. We then pay in \$500, leaving \$3510.01 in the account. The next month, the debt in the account would grow to $3510.01e^{.03(1/12)}$ (note that we used 1/12 again, as this is the amount of time (in years) that passed from this month to the next. This is about \$3519.80. We again pay in our \$500, and we keep on going. We see that on the 9th month, we won't need all \$500. So we use what we need, and then put the next in the next-smallest account. How big is that account now? Looking above, we see it had interest rate 3.5%. After 9 months, it will have grown to size $4000e^{.035(9/12)}$, or about 4091 dollars. Continue in this fashion, paying off the different debts in this order. How long does it take, and what is the total cost?

(3) Now, let's use a better scheme. Pay off the largest interest rates first. How long does it take, and what is the total cost?

(4) Now, I give you an option, Either come up with your own payback method to try, or do the following computationally intense method - each month, pay the interest on all accounts, and with the leftovers, pay off the highest account. This isn't actually much harder or longer, once you realize that the interest payment on all but one loan don't change from month to month.

(5) Which of these is the best way to pay off one's debt? Note that in every case, there's an interesting property: it's hard to make progress at first, as there is something like \$100 in interest each month. But as you pay more off, the interest rates fall, and it gets easier. This intuition messes with a lot of people's finances. This also leads to the wisdom that large initial payments reduce overall pain by a lot.

(6) *This is very similar to the financial situation one of my friends found themselves in, except the numbers were not this clean. He got an engineering position, but he worked as a waiter for 2 months as*

well to supplement his initial payments. Those 2 months ended up reducing the length of his payment period by about 8 months.

It's always scary when an exercise takes more than a page to write down, hmm?

2.5. **Trigonometric functions.** It goes without saying that you should know the unit circle, at least the $\sin x$ and $\cos x$ values. And you should feel comfortable with both degrees and radians, and know how to convert between them.

Exercise 2.30. Convert the following angles from degrees to radians:

(1) $20°$
(2) $45°$
(3) $110°$
(4) $75°$

Exercise 2.31. Convert the following angles in radians to angle measures in degrees:

(1) $\pi/12$
(2) $\pi/6$
(3) $5\pi/7$
(4) $12\pi/13$

Example 2.32. We have two fundamental ways of thinking about the elementary trig functions, $\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta, \cot\theta$. On the one hand, they describe the relationship between the adjacent, opposite, and hypotenuse sides of a right triangle. On the other hand, $\sin\theta$ describes the y-coordinate of the point on the unit circle at an angle θ with the positive x axis, and $\cos\theta$ describes the x-coordinate. The other functions can all be determined from the sine and cosine values.

Example 2.33. Determine the values of all 6 elementary trigonometric functions from the given information:

(1) $\theta = \pi/6$
(2) $\theta = -\pi/3$
(3) $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta < 0$
(4) $\tan\theta = 1, \sin\theta < 0$
(5) $\cot\theta = -\sqrt{3}, \sin\theta > 0$

Example 2.34. Not covered here, but review the graphs of the six elementary trig functions, and their periods. One thing that I undercovered in class was horizontal shifting of trig with modified periods. For example, suppose we looked at $g(x) = 3\sin(2x - \pi)$. The period is π, which is half the normal period, due to the coefficient 2 in front of the x. The amplitude of this function is 3, as it's the leading coefficient. How much is the function shifted by? It is **not** shifted by π. Instead, note that this function is $3\sin(2(x - \pi/2))$, so that in fact we see that the function is shifted to the right by $\pi/2$ (in fact, if the function $h(x) = 3\sin(2x)$, then we see easily that $g(x) = h(x - \pi/2)$). The period change and the shift change might confuse answers. An easy way to check is to use the most basic fact about sine: $\sin(0) = 0$. Thus if we think it's shifted to the right by $\pi/2$, then we would expect $g(\pi/2) = 0$. This is the case. Is $g(\pi) = 0$?

3. Analytic Trigonometry

We now transition away from the simple aspects of the elementary trigonometric functions, like what they look like and their basic manipulations. We spend a whole 2 weeks on trig, equating to one week of 'elementary' trig and one week of 'analytic' trig.

To review, we more or less covered the law of sines, the law of cosines, completing triangles knowing 1 side length and any other 2 pieces of information (from the other two side lengths or the three angles). We discovered what combinations of side lengths/angles are or are not possible for triangles to have, and we learned how to dea with possibly ambiguous descriptions of triangles (e.g. if two side lengths and a non-included angle are known, two resulting triangles may be possible).

From there, we learned the angle-sum and angle-difference formulas for sine and cosine, and saw that these gave rise to a great host of other formulae. Some worth mentioning might be the sum-to-product, product-to-sum, and power reduction formulas. There is an incredible amount of power behind these simple tools, and being able to tap into them is a big goal, and the primary purposeof this section.

As a side-note, we came at this material through complex numbers. This will serve as the reference write-up for the complex-number approach as well.

3.1. **Laws and Identities.** Within this course, I will never expect you to simple have everything memorized. You will always be permitted to have a cheat-sheet when things like trig identities come up. In your study habits, you shouldn't focus on memorizing all the different formulae, but instead focus on learning when to apply each one. Drawing pictures, triangles, and the unit circles are probably good habits to get into here.

Example 3.1. The Law of Sines says that for any triangle ABC with sides a, b, c, such that A is the angle opposite a, then $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. This is one of the trig laws that is easy to prove is you forget it (recall from class, on a triangle, you drop an altitude, and compute its length in two different ways by using that $\sin(\theta) = \frac{o}{h}$ and set them equal). The Law of Cosines is harder to remember. It says that $a^2 = b^2 + c^2 - 2bc\cos A$. (To remember how we proved it - we used the distance formula and the Law of Sines, and it just sort of popped out).

Example 3.2. The primary purpose of these laws is to complete triangles when we know only a little about them. It turns out that if we know any side length and any other 2 bits of information, we can classify the triangle. However, not every combination of sides and angles are possible. For example, there is no triangle for which $a = 10, b = 20, A = 85°$. Draw what this triangle would need to look like, and you can see why. Algebraically, if we try to solve for $\sin B$ with the Law of Sines, it explodes. Similarly, there is no triangle with side lengths $a = 1, b = 2, c = 4$. Trying to draw one would make this very obvious again.

Example 3.3. On the other hand, it's possible for there to be two triangles that both have the same SSA. For example, there are two triangles with $a = 12, b = 31, A = 20°$. Draw a picture, and the key bit here is to remember that there is an acute and an obtuse triangle that fits. The trickiest part about this type of triangle is that if you naively compute it using your calculator, you'll only get one of the two triangles. This is because you will want to use the Law of Sines at one point, and you'll have to find the inverse sine of some angle - suppose you were finding $\arcsin 0.9047$, ad your calculator would spit out $64.8°$, but $180° - 64.8° = 115.2°$ is another angle whose sine is 0.9047. Why is this? It has to do with our convention on $\arcsin(x)$, making it so that it's single-valued. So it's important to draw a picture and think about what you're doing.

Exercise 3.4. Solve the following triangles:

(1) $A = 36°, a = 8, b = 5$
(2) $A = 110°, a = 125, b = 100$
(3) $A = 58°, a = 15, b = 17$
(4) $a = 6, b = 8, c = 12$
(5) $A = 50°, b = 15, c = 30$
(6) $C = 15°, a = 6.25, b = 2$

Example 3.5. The area of a triangle is given by $\frac{1}{2}bh$. In class, we came up with a method of finding the area of any triangle using the Law of Sines and the Law of Cosines. But the fact is that it's not so hard - on any triangle, drop an altitude so that we have a right triangle with known angle and hypotenuse. Then use $\sin\theta = \frac{o}{h}$ to find the length of the altitude. This is the rationale that we used to show that Area $= \frac{1}{2}bc\sin A$, for example.

Exercise 3.6. For each of the triangles in exercise 3.4 above, find their area.

Exercise 3.7. There are some other identities that we should know. Justify why the following are true:

(1) $\sin^2 x + \cos^2 x = 1$
(2) $1 + \cot^2 x = \csc^2 x$
(3) $\tan^2 x + 1 = \sec^2 x$

The sine and cosine angle-sum and angle-difference formulas are very important, and will provide our primary tools for simplifying or verifying trigonometric expressions.

Example 3.8. The sine angle-sum formula says that $\sin\alpha+\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and the cosine angle-sum formula says that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. From these, we can remember all the others if we have enough time. To get the angle-difference formula, we must remember that $\sin(-x) = -\sin(x)$, i.e. that sine is an odd function, and that $\cos(-x) = \cos(x)$, i.e. that cosine is an even function. For then $\sin(x-y) = \sin(x+(-y)) = \sin x\cos(-y) + \sin(-y)\cos x = \sin x\cos y - \sin y\cos x$. Similarly, $\cos(x-y) = \cos x\cos y + \sin x\sin y$.

Example 3.9. One of the most fundamental things we can use angle-sum and difference formulae for is to evaluate the exact values of some angles that we don't already know. We know the values for all multiples of 30° and 45°. What if we wanted the exact value of 75°? We can note that $75° = 30° + 45°$, so that $\cos 75° = \cos(45° + 30°) = \cos(45°)\cos(30°) - \sin(45°)\sin(30°)$. We know the exact values for all of these values, so we can simply multiply them out to get that $\cos(75°) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.

Exercise 3.10. Find the exact values of the following:

(1) $\cos(115°)$
(2) $\sin(135°)$
(3) $\tan(225°)$
(4) $\sin(15°)$
(5) $\sin 105°)$ *remember what you've already calculated*

Example 3.11. This example describes a type of problem that **will absolutely come up in calculus, in something called trig substitution**. Suppose we wanted to write $\cos(\arctan 1 + \arccos x)$ algebraically, knowing that $0 \leq x \leq 90°$. Then we can note that $\cos(\arctan 1 + \arccos x) = \cos(\arctan 1)\cos(\arccos x) - \sin(\arctan 1)\sin(\arccos x)$. We know that $\arctan(1) = 45°$, so that $\sin(\arctan(1)) = \cos(\arctan(1)) = \frac{1}{\sqrt{2}}$. How do we evaluate the other two terms? We know that $\cos(\arccos x) = x$ for x in $[0, 90°]$ (*but not all x - do you remember why?*). To know $\sin(\arccos x)$, we need to find a triangle that has an *arccosx* angle. So we choose the right triangle with hypotenuse length 1 and adjacent side-length x. Then the enclosed angle is $\arccos x$. What's the remaining side-length? It's $\sqrt{1^2 - x^2}$ by the Pythagorean Theorem. Now it's easy to see that $\sin\arccos x = \sqrt{1-x^2}$. Putting this all together, we see that $\cos(\arctan 1 + \arccos x) = \dfrac{x - \sqrt{1-x^2}}{\sqrt{2}}$.

Exercise 3.12. Write the following algebraically:

(1) $\sin(\arccos \frac{1}{2} + \arccos x)$
(2) $\cos(\arctan\sqrt{3} + \arcsin x^2)$
(3) $\sin(\arccos \frac{3}{5} - \arcsin \frac{5}{13})$ *(This doesn't require a calculator - and you might even give a cleaner answer than your calculator)*

Exercise 3.13. Solve the following for x in $[0, 2\pi)$ *(The theme here is that if you can simplify trig expressions so that the angles aren't shifted, then perhaps you should do it)*:

(1) $\sin(x + \pi/3) + \sin(x - \pi/3) = 1$
(2) $2\sin(x + \pi/2) + 3\tan(\pi - x) = 0$

Let's guide you through some verifications of other trigonometric identities using the few that we know so far.

Exercise 3.14. Let's prove the double-angle formulae: $\sin(2u) = 2\sin u \cos u$ and $\cos(2u) = \cos^2 u - \sin^2 u$.

(1) Interpret $\sin(2u)$ as $\sin(u + u)$ and use angle-sum.
(2) Do the same for $\cos(2u)$.
(3) Let's write $\cos(2u)$ as $2\cos^2 u - 1$. Do you see how to do that?
(4) Now let's write $\cos(2u)$ as $1 - 2\sin^2 u$.

Exercise 3.15. Show the power-reduction formulae: $\cos^2 u = \dfrac{1 + \cos 2u}{2}$ and $\sin^2 u = \dfrac{1 - \cos 2u}{2}$. *(Hint: use the last exercise)*.

Example 3.16. We can use the double-angle formulas for larger angles, too. If we wanted to come up with the sine triple-angle formula, we might note that $\sin 3x = \sin(x + 2x) = \sin x \cos 2x + \sin 2x \cos x = \sin x(\cos^2 x - \sin^2 x) + (2\sin x \cos x)\cos x = 3\sin x \cos^2 x - \sin^3 x$.

Exercise 3.17. Find the cosine triple-angle formula and quadruple-angle formula. *(Hint: I'd recommend breaking up $4x$ as $2x + 2x$)*.

Exercise 3.18. Let's put these skills to some use. Given that $\cos\theta = 5/13$, $0 < \theta < \pi/2$, find $\sin 2\theta, \cos 2\theta, \tan 2\theta$.

Exercise 3.19. Let's derive the half-angle formulas $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$ and $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$.

(1) Start with the power-reduction formulae from exercise 3.15 above.
(2) Write the variable $x = 2u$, and write the power-reduction formulae in terms of x instead of u.
(3) Solve for the squared trig term in the power-reduction formulae. This should give you the result.

Exercise 3.20. Now we deviate from the process of using our last set of identities to come up with a new set. Verify some of the product-to-sum formulas.

(1) Show that $2\sin u \sin v = \cos(u - v) - \cos(u + v)$
(2) Show that $2\cos u \cos v = \cos(u - v) + \cos(u + v)$

3.2. **Putting all the analytic trig together.** For this section, I highly recommend you either work with the book's page open to the trig identities, or after having written a trig cheat sheet (like the one you'll be allowed to use on the final, for example).

Exercise 3.21. Subtract and simplify: $\dfrac{\sin\theta - 1}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta - 1}$

Exercise 3.22. Verify the identities
(1) $\cot^2\alpha(\sec^2\alpha - 1) = 1$
(2) $\sin^2 x\cos^2 x = \frac{1}{8}(1 - \cos 4x)$
(3) $\dfrac{2\cos 3x}{\sin 4x - \sin 2x} = \csc x$

Exercise 3.23. Solve the following for x in $[0, 2\pi)$:
(1) $3\tan x - \cot x = 0$
(2) $\cos^2 x + \cos x = 0$
(3) $\sin 2x - \cos x = 0$

4. Complex Numbers and Vectors

We review the basic manipulation of complex numbers and their interaction. Understanding 2D vectors and complex numbers are very similar, so we reconsider them together.

4.1. **Basics of Complex Numbers.** In many ways, the complex numbers behave just as we suspect. Any complex number z can be written uniquely as $a + bi$, where a, b are both real numbers, and $i = \sqrt{-1}$. If z_1, z_2, z_3 are three complex numbers, then we know that

- $z_1z_2 = z_2z_1$ (commutativity)
- $(z_1z_2)z_3 = z_1(z_2z_3)$ (associativity)

For a complex number $z = a + bi$, we call a the 'real part' and b the 'imaginary part.' If $z_1 = a + bi, z_2 = c + di$, then $z_1z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

To each complex number $z = a + bi$, we can associate the 2D real vector (a, b). This means that the more we know about 2D real vectors, the more we know about complex numbers, and vice versa. Recall that to a 2D real vector $\vec{u} = (x, y)$, we denote by $|\vec{u}|$ the magnitude of $\vec{u}$ (which is $\sqrt{x^2 + y^2}$, from the Pythagorean Formula), and an angle θ that the vector makes with the positive x axis (thus $\theta = \arctan y/x$). Similarly, for a complex number $z = a + bi$, we have $|z| = \sqrt{a^2 + b^2}$ and $\arg(z) = \theta$ (we call the complex angle the argument).

Exercise 4.1. Perform the following manipulations with complex numbers, writing the answer if the form $a + bi$):
(1) Compute $(3 + 4i)(5 + 6i)(1/2 + (3/4)i)$
(2) Compute $(1 + i)^3$
(3) If $z = 8 + 11i$, what is $|z|$? What is $\arg(z)$?

Note that in this last exercise, you would do the same work to find the length and angle of the vector $(8, 11)$. In this sense, these are just two sides of the same coin. More basic exercises on complex numbers can be found in the first set of exercises I put up, after the first week. But we won't focus on complex numbers here. Our main purpose is to remember the idea of our proof of the angle-sum formulae.

4.2. **Complex proof of angle-sum.** Recall that we can write any 2D real vector $\vec{u}$ as $|u|(\cos\theta, \sin\theta)$, where θ is the angle that the vector makes with the positive x axis. Similarly, any complex number z can be written as $|z|(\cos\theta + i\sin\theta)$, where θ is the argument of z.

The only result for which we gave no justification throughout this course was the following:

Theorem 4.1. ***Euler's Theorem:*** *The following stunning theorem is true, relating the magnitude and angle of a complex number, the exponential, and trig functions:*

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.1}$$

Example 4.2. This means that we know that $e^{i(2\pi/3)} = \cos(2\pi/3) + i\sin(2\pi/3) = -1/2 + i\sqrt{3}2$, for example.

Lemma 4.2. *Theorem 4.1 lets us do some very slick things. One case is that*

$$(\cos\theta + i\sin\theta)^n = e^{(i\theta)n} = e^{i(\theta n)} = \cos(n\theta) + i\sin(n\theta) \tag{4.2}$$

This is a trigonometric statement that we proved with the geometry of complex numbers. Cool.

Example 4.3. This also lets us perform some otherwise hard computations very easily. Since every complex number can be written as $r\cos\theta + i\sin\theta$, where I use a standard polar notation $r = |z|$ (recall polar coordinates), then we also have that every complex number can be written as $re^{i\theta}$. Suppose we wanted to calculate $(1+i)^{17}$. We could do this by hand, but that's painful. Instead, note that $1+i = \sqrt{2}(\cos\frac{\pi}{2} + i\sin\frac{pi}{2})$ (you should be able to do that - can you?), so that $1+i = \sqrt{2}e^{i\pi/2}$. Thus $(1+i)^{17} = \sqrt{2}^1 7e^{i17\pi/2} = \sqrt{2}^1 7e^{i\pi/2}$ (it's periodic with period 2π, so we can cancel out multiples of 2π), which is $\sqrt{2}^1 7(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2}^1 6(1+i) = 256 + 256i$. Whoa.

Now we proceed to our main proofs. I present these because it's how most people I know remember the angle-sum identities.

Lemma 4.3. *Proof of Angle-Sum Formulae:*

$$\sin(x+y) = \sin x\cos y + \sin y\cos x \tag{4.3}$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \tag{4.4}$$

Proof: *Consider the complex numbers* $z_1 = \cos x + i\sin y = e^{ix}$ *and* $z_2 = \cos y + i\sin y = e^{iy}$. *Multiply them together. On the one hand,* $(\cos x +$

$i \sin x)(\cos y + i \sin y) = (\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \sin x \cos y)$. *On the other hand,* $e^{ix}e^{iy} = e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$.

Two complex numbers are the same if and only if they have the same real and complex values. Thus, since we have $(\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \sin x \cos y) = \cos(x+y) + i \sin(x+y)$*, we must have that* $\cos(x+y) = (\cos x \cos y - \sin x \sin y)$ *and* $\sin(x+y) = \cos x \sin y + \sin x \cos y$*, proving our two formulae. And think about it - this is pretty easy to remember, as long as we remember that* $e^{i\theta} = \cos\theta + i \sin\theta$*.* $\diamondsuit$

4.3. **Vector Arithmetic.** *We also developed 2D vector arithmetic. One important skill we developed was the ability to write any vector in the form* $(r\cos\theta, r\sin\theta)$*, which we later converted to polar form* (r, θ)*.*

Example 4.4. Suppose we wanted to write the vector $(1, \sqrt{3})$ in trigonometric-vector form $(r\cos\theta, r\sin theta)$. Then we compute the length to be $\sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$, and the angle to be $\arctan\sqrt{3} = \pi/3$. Thus $(1, \sqrt{3}) = (2\cos\pi/3, 2\sin\pi/3)$, and a quick computation shows this is correct. Thus if we were to write $(1, \sqrt{3})$ in polar coordinates, its polar coordinates would be $(2, \pi/3)$.

Exercise 4.5. Write the following vectors in trigonometric-vector form and in polar coordinates:

(1) $(5, 5)$
(2) $(3, 4)$
(3) $(7, 11)$

We also learned dot products. We learned a lot of material with respect to dot products, and this will be used a lot in physics (though not AP Calculus BC, due to its course design). Do you remember what a dot product is?

Exercise 4.6. Compute the following dot products:

(1) $(1, 3) \cdot (2, 4)$
(2) $(2, 3) \cdot (-3, 2)$ (What does this result mean about the two vectors?)
(3) $(4, 4) \cdot (1, -1/2)$

Recall that we also used dot products to project vectors onto other vectors. This will come up the next time you learn about vectors, in particular when you learn a topic called "Linear Algebra." In this course (and in your first physics course), you can actually get by using only triangles and properties of sine, cosine, and tangent. To that end, *there will be no dot product projections on the final exam, nor (unless I'm very mistaken) on a placement exam into calculus.*

5. Conics

I feel this is recent enough that I don't need to go through examples to review. What should you know? You should know the standard forms of the four conics, how to draw them, and remember the general form of a conic. This is in line with a big motif of this course: any time you see a quadratic, you should feel absolutely confident in your ability to solve it and understand it.

Also - what is the key to putting conics in standard form? *Ans: completing the square.* So we do 2 circles, 2 ellipses, 2 parabolas, and 2 hyperbolas.

Exercise 5.1. Put the following conics in standard form, state what conic they are, and graph them.

(1) $x^2 - 6x + y^2 + y - 90.75 = 0$
(2) $9x^2 - 32x - 9y^2 + 36y - 8 = 0$
(3) $y^2 - 2y - 8x = 7$
(4) $4x^2 + 3y^2 - 8x + 6y - 5 = 0$
(5) $x^2 + y^2 + 6y + 8 = 17$
(6) $16x^2 + 16x + 4y^2 - 4y = 59$
(7) $x^2 + 13 = 4y - 6x$
(8) $2y^2 - x^2 - 4y = 2$

You will be allowed a 'cheat sheet' for the final exam. On this sheet, you may put anything that you'd like. I don't expect you to have the various bits about the conics memorized, like how to find the focus of a parabola from its formula. But since you can have a cheet sheet, I would feel no guilt if I asked you to do such a thing on the final. So be able to do it with a sheet, even if you can't do it from memory.

This, I think, is a perfectly valid skill. We focus so often on memorization, when it's really the ability to quickly synthesize different sources and information that matters. This does mean quickly, however, so you must have some familiarity with the skill. Like all other tests so far, the final will test your speed and familiarity with the material as well as your understanding.

6. Sequences and series

We haven't finished this yet. Coming Tuesday afternoon.

7. Limits and an Introduction to Calculus

We haven't finished this yet either. Coming Tuesday/Wednesday afternoon.