REVIEW EXERCISES

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Contents

1. Introduction

We have now covered three-fifths of the material from this course. While in many ways, this course has been cumulative and we have revisited much of the earlier material repeatedly, there are things that have been left out. In this packet, we will review some of the problem-types that we have come across and methods of finding their solution.

We will also take this time to combine our new skills, such as our knowledge of trigonometric identities and transcendental functions, with some of our old skills.

2. Elementary Functions

We started by reviewing basic factoring and graphing linear equations. We then worked on developing the algebraic and graphical qualities of polynomials. We learned to solve any quadratic equation that we will ever see (any cubic too, though that's much harder). Later came the Factor Theorem and the Rational Root Theorem, allowing us to solve more. This all culminated with rational functions. This first set of exercises covers this material.

2.1. Factoring and Solving Quadratics.

Example 2.1. One tool that we use with high frequency is factoring. We cannot get away from factoring, it turns out. Many tools revolve around factoring. We'll look at three major factoring tools here.

(1) $(x+y)^2 = x^2 + 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$

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(2) $(x-y)(x+y)=x^2-y^2$
(3) Completing the square

Example 2.2. We can use $(x+y)^2 = x^2 = 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$ both ways. For example, when we see an expression that we'd like to expand, like $(\sin x + \cos x)^2$, we can immediately say that it is $\sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + 2\sin x \cos x$ without any FOILing. (further, if we are exceptionally clever, we might remember that $2\sin x \cos x = \sin(2x)$).

On the other had, we can factor functions quickly too. For example, when we are asked to find the roots of $e^{2x} - 4e^x + 4$, we recognize this as $(e^x - 2)^2$. Thus there is a double root at $x = \ln 2$.

Exercise 2.3. Expand the following without directly multiplying it out:

- $(3x+2y)^2$
- $(5x-4)^2$
- $(x+\cos x)^2$
- $(\sec x - \sin x)^2$
- $(e^x + e^{-x})^2$

Exercise 2.4. Factor the following:

- $\cos^2 x + 2\cos x \tan x + \tan^x x$
- $2e^{2x} + 6\sqrt{2}e^x + 9$
- $\sin^2 x + 8\sin x + 16$
- $2 + x^2 + x^{-2}$

Example 2.5. It is usually very easy to see cases where we have $(x-y)(x+y)$ and rewrite it as $x^2 - y^2$, but we sometimes need to approach it in the opposite direction. For example, if we want to find the roots of $\sin^2 x - 1/2$, we can do this quickly and easily with this factoring method. $\sin^2 x - 1/2 = (\sin x - 1/\sqrt{2})(\sin x + 1/\sqrt{2})$, and thus the solutions are $x = \pi/4 + n\pi/2$.

Exercise 2.6. Factor the following:

- $3x^2 - 2y^2$ (just because everything starts as 'squares' doesn't mean that they're the squares of pretty numbers)
- $4e^{2x} - 9$
- $2\tan x - 4$

Further, if we want to factor over the complex numbers, we might notice that $x^2 + y^2 = (x+iy)(x-iy)$. So we cam factor the following:

- $4x^2 + 9y^2$
- $2x^2 + 16y^2$

Example 2.7. There is a general form for completing the square that always works. If we have $x^2 + ax + b$, we can note that $x^2 + ax + \frac{a^2}{4} - \frac{a^2}{4} + b = (x+\frac{a}{2})^2 - \frac{a^2}{4} + b$. For example, $x^2+3x+5 = x^2+3x+\frac{9}{4}-\frac{9}{4}+5 = (x+\frac{3}{2})^2-\frac{9}{4}+5$. If we believe this pattern, we could skip the middle.

For example, if we had $x^2+10x+3$, we could write this as $(x+\frac{10}{2})^2-25+3$ without any of the middle expansions.

Exercise 2.8. Complete the square on the following:

- $x^2 + 18x + 2$
- $x^2 + 5x + 3$
- $\sin^2 x + 3\sin x + 2$
- $2x^2 + 3x + 5$ (the task here is to remember how to deal with the leading 2)

The goal is for these factoring techniques to feel like second nature. The less one needs to think about them, the better. The task of finding roots is largely equivalent to factoring, due to the Factor Theorem. In short, this says that if $p(x)$ is a polynomial and $p(r) = 0$, then $p(x) = (x-r)q(x)$ where $q(x)$ is a smaller degree polynomial. Thus we can pull out a linear factor of $(x - r)$.

Example 2.9. Anytime we see a quadratic, we should be happy. We can solve quadratics, always. Through factoring or the quadratic formula, we can always solve quadratics. In fact, we can solve them quickly. One shouldn't need to spend more than a minute on a quadratic in the form $ax^2+bx+c = 0$.

Exercise 2.10. Solve the following quadratics:

- $x^2 + 5x + 18 = 0$
- $4x^2 + 3x + 2 = 0$
- $2\sin^2 x + 3\sin x + 1 = 0$
- $3\cos^2 x + 8\sin x + 1 = 0$
- $2e^{4x} - 13e^{2x} + 1 = 0$

In cases (like above), remember that $\cos^2 x + \sin^2 x = 1$, and this can be modified to relate $\csc^2 x$ to $\cot^2 x$, or to relate $\sec^2 x$ to $\tan^2 x$. It's important to recognize 'hidden quadratics,' and to do the necessary work to transform a quadratic into a form that you can solve.

Exercise 2.11. Solve the following quadratics:

- $5\cot^2 x + 14\csc x + 1 = 0$
- $2\cos^2 x + 4\sin x + 2 = 0$
- $\tan^2 x + 5\sec x + 3 = 0$
- $(x - 2)^2 + 2x + 4 = (x - 1)$
- $(x - 3)^3 + (x - 1)(x + 1) = x^3 + 1$
- $\sqrt{x-2} + \sqrt{x+2} = 2$
- $\sqrt{x+1} + \sqrt{x-4} = \sqrt{x} + 5$

2.2. **Polynomial Inequalities.** Perhaps the best way of solving polynomial inequalities is to find its roots, make a sign chart, and just test on each side of each root. It is almost certainly the fastest. This is our general method:

Example 2.12. When we see a polynomial inequality $p(x) \geq 0$, we find the roots $r_1, \ldots, r_n$ of the polynomial. We then draw a number line, and see if the polynomial is positive or negative between each pair of roots. As polynomials are continuous, they will only change signs at roots. We use this to decide on our inequality. As an aside, I want to mention that this is one of the easiest types of questions to merit partial credit as long as you show your work, if you're in a graded situation.

Example 2.13. Let us solve the quadratic inequality $3x^2 + 4x \geq 1$. First, we gather ever everything to one side. So we want to solve $3x^2 + 4x - 1 \geq 0$. Where are the roots of the quadratic $3x^2 + 4x - 1$? This doesn't immediately seem to factor nicely, so we use the quadratic formula: the roots are $x_+, x_- = \dfrac{-4 \pm \sqrt{16+12}}{6}$. These roots split the real line into the three regions $(-\infty, x_-), (x_-, x_+), (x_+, \infty)$. We need to check the sign of our polynomial on each of the three regions. Using, for example, $-100, 0, 100$, we see that the signs go $+ - +$. Thus the inequality's solution is x in $\left(-\infty, \dfrac{-4-\sqrt{28}}{6}\right]$ and $\left[\dfrac{-4+\sqrt{28}}{6}, \infty\right)$.

Exercise 2.14. Solve the following quadratic inequalities.

- $8x^2 - x > 3$
- $9x^2 + 6x > 1$
- $4x^2 \geq 4$

The same idea works for higher degree polynomials as well. The task is the same: find the roots, make a number line, identify regions where the polynomial is positive and negative, and use this to find your answer. Also remember - **it is not always the case that the sign switches positive negative positive negative.**

Exercise 2.15. Solve the following polynomial inequalities. These can be done with factoring.

- $x^2 + 4x - 6 \geq 6$
- $2x^2 + x - 15 < 0$
- $-x^2 + 2x + 3 \geq 0$
- $x^3 - x^2 - 16x + 16 \leq 0$
- $x^3 - x^2 - 16x + 16 \geq 36$
- $x^4 - x^2 - 20 > 0$

Exercise 2.16. Solve the following polynomial inequalities. You may have to use other tools, such as the rational root theorem or factor theorem, to proceed here.

- $x^4 - x^3 - 2x - 4 > 0$
- $x^5 - x^4 - 3x^3 + 5x^2 - 2x$

2.3. **Rational Functions.** In many ways, understanding rational functions comes down to understanding polynomials. Once we understand polynomials and, in particular, identifying where they are positive, negative, or zero, we know a tremendous amount about rational functions.

The general method of attacking rational functions is to find the zeroes of the numerator and denominator, set up a sign chart with these zeroes as the important places, and to identify where the rational function will be positive and where it will be negative. Zeroes of the numerator lead to zeroes of the rational function. Zeroes of the denominator lead to vertical asymptotes of the rational function. If there is the same zero in the numerator and denominator, then there might be a hole.

The only bit remaining with respect to rational functions is to understand their limiting behavior. This falls into a few different categories: there might be a horizontal asymptote, a slant asymptote, or no asymptote.

Example 2.17. Consider the rational function $f(x) = \dfrac{x^2+3x+5}{x^3+x+1}$, and suppose we want to find its limiting behavior. If we think of really large x, then x^3 is much larger than x^2. In general, if the degree of the denominator is greater than the degree of the numerator, than the limiting behavior is a horizontal asymptote at $y = 0$. That is the case here.

Example 2.18. Consider the rational function $g(x) = \dfrac{x^3+3x+1}{4x^3+1}$, and suppose we want to find its limiting behavior. If we think of really large x this time, we can't use the same trick as above. Now the degree of the numerator and denominator are the same. But for really large x, everything except the x^3 and $4x^3$ terms matter less and less. In general, if the degrees of the denominator and numerator are the same, then there is a horizontal asymptote. For $g(x)$, we expect $g(x) \approx \frac{1}{4}$ for really large x, as the x^3 term of the numerator gets divided by $4x^3$ in the denominator. This leads to the general fact that the horizontal asymptote in these cases will be at $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.

Example 2.19. Consider the rational function $h(x) = \dfrac{x^3+3x+1}{x^2+1}$. The degree of the numerator is exactly one more than the degree of the denominator. Using polynomial long division, we see that $h(x) = x + \dfrac{2x+1}{x^2+1}$, so that for large x the polynomial behaves just like x (the $\dfrac{2x+1}{x^2+1} \to 0$ as x gets big). We call the line x in this case the *slant asymptote*, and we find it in general by performing polynomial long division.

Example 2.20. Consider the rational function $j(x) = \dfrac{x^5+3x^2+1}{x^2+1}$. Polynomial long division would reveal limiting behavior similar to a cubic, as the degree of the numerator is 5 and the degree of the denominator is only 2. We don't care about 'curved' asymptotes in this course, so all that we care about here is whether the function goes to ∞ or $-\infty$ as $x \to \infty$ and $x \to -\infty$.

Exercise 2.21. Find the zeroes, vertical asymptotes, holes, horizontal asymptotes, and slant asymptotes of the following rational functions. Sketch the results.

(1) $\dfrac{x^2 - 5x + 4}{x^2 - 4}$
(2) $\dfrac{2x^2 - 5x + 2}{4x^2 - 2x - 12}$
(3) $\dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$
(4) $\dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 4x + 2}$ (similar to, but not the same as, the previous)

Exercise 2.22. Let's see a sort of way in which rational functions might come up. Certain professions, such as any sort of manufacturing or chemical engineer, need to worry about particular types of problems that we call "mixing problems." Suppose, for instance, that a large tank contains 50 liters of a 75%/25% water/sodium benzoate solution. We want a larger concentration of sodium benzoate, but it's challenging and expensive to get pure sodium benzoate. But it's easy to get a 75%/25% sodium benzoate/water mixture. So we pour x liters of this new mixture into the tank.

- Show that the new concentration C (starting at 0.25 and changing because we are adding liquid with a 0.75 concentration) is given by $C = \dfrac{3x + 50}{4(x + 50)}$
- Find the limiting behavior of this system (for positive x only - the implied domain of this model is for x positive only. Why is that?).
- Does this limiting behavior make sense?

In fact, mixing problems are very important. But to be fair, this would be one of the easiest mixing problems out there. Chemical engineers, for example, have to work with different concentrations of different materials interacting with each other - and different concentrations change the rate of chemical reaction and interaction as well. There is some intense math there - but this is where it starts.

Exercise 2.23. In my work, I happen to use rational functions quite a bit. There are some miraculous properties of rational functions. For better or worse, we look at two of them here.

(1) Often, math asks meta-type questions: instead of "what is the solution?" it might ask "when is this solveable?" For example, for what k is the equation

$$x^2 + (1 - 3k)x + (2 - k) = 0 \tag{2.1}$$

solveable for real-valued x? *To do this, 'solve' for k. You'll get a rational function. Find the range of that rational function, and this will be the exact vales of k for which equation (2.1) is solveable.*

(2) This introduces a surprising relationship between rational functions, matrices, and complex numbers. Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can associate a rational function $f(z) = \dfrac{az+b}{cz+d}$ on the complex numbers. There are some stunning things here: if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then the rational function $f(z) = \dfrac{az+b}{cz+d}$ is invertible with inverse $f^{-1}(z) = \dfrac{ez+f}{gz+h}$. This is not at all obvious, and is a bit surprising. If you recall the geometry of complex numbers, and remember that multiplying has to do with a certain rotation and scaling operation, then one can view the associated rational functions to matrices as doing a certain rotation, scaling, shifting, and then doing another roation, scaling, and shifting. The work I do uses this sort of interplay extensively, and this hints at two pervasive concepts of higher mathematics: we find connections between different objects, ultimately learning more about everything involved; and we let things 'act' on other things (in this case, matrices are 'acting' on the complex plane) and through these actions, we learn more about both what's being acted upon and the actor.

2.4. Exponentials and Logs.

Exercise 2.24. Review the basic definitions and properties of exponentials and logarithms. Also review the change-of-base formula.

Example 2.25. Our key interest with exponentials was with modeling certain types of growth. The easiest to remember is compounding interest. If an initial payment of P is put into an account that grows as r percent interest that compounds n times a year, then after t years, there will be $P(1 + \frac{r}{n})^{nt}$ in the account. If the interest compounds continuously, there will be Pe^{rt} in the account.

Exercise 2.26. Find the amounting of money in the accounts at the given amount of time shown:

(1) placeholder

Exercise 2.27. Let's do an experiment. Suppose you are in college debt, a situation which forces some to get a new loan every 6 months for 4 to 5 years. After some amount of time, you might have to pay back 9 or so different loans, each with their own interest rates. Think to yourself about the following: what's the best way to pay it back? Choose the largest interest account and pay that one off? Distribute money across several accounts? Pay off the interest on each, but focus on one or another? *This exercise will be a bit computation heavy, so I recommend that you pull out your calculator,* *some paper, and keep great notes and a table. It is these notes/table that I'll want to see*

(1) Let us suppose each of the 9 loans is for 4000 dollars, and they have the following anual interest rates: $3\%, 3.5\%, 4\%, 4.4\%, 4.8\%, 5\%, 5.2\%, 5.2\%, 5.4\%$, compounding continuosly (it will give a good approximation). Let us also suppose that we have 500 dollars available per month to pay into these loans, and we invest these at the end of the month, each month.

(2) First, let's see what happens if we use a dumb payback scheme: pay off the smallest interest first, and then progress higher. The smallest loan debt would grow like $4000e^{.03t}$. After one month, the debt on this account would be $4000e^{.03(1/12)}$. This is about \$4010.01. We then pay in \$500, leaving \$3510.01 in the account. The next month, the debt in the account would grow to $(3510.01e^{/03(1/12)}$ (note that we used 1/12 again, as this is the amount of time (in years) that passed from this month to the next. This is about \$3519.80. We again pay in our \$500, and we keep on going. We see that on the 9th month, we won't need all \$500. So we use what we need, and then put the next in the next-smallest account. How big is that account now? Looking above, we see it had interest rate 3.5%. After 9 months, it will have grown to size $4000e^{.035(9/12)}$, or about 4091 dollars. Continue in this fashion, paying off the different debts in this order. How long does it take, and what is the total cost?

(3) Now, let's use a better scheme. Pay off the largest interest rates first. How long does it take, and what is the total cost?

(4) Now, I give you an option, Either come up with your own payback method to try, or do the following computationally intense method - each month, pay the interest on all accounts, and with the leftovers, pay off the highest account. This isn't actually much harder or longer, once you realize that the interest payment on all but one loan don't change from month to month.

(5) Which of these is the best way to pay off one's debt? Note that in every case, there's an interesting property: it's hard to make progress at first, as there is something like \$100 in interest each month. But as you pay more off, the interest rates fall, and it gets easier. This intuition messes with a lot of people's finances. This also leads to the wisdom that large initial payments reduce overall pain by a lot.

(6) *This is very similar to the financial situation one of my friends found themselves in, except the numbers were not this clean. He got an engineering position, but he worked as a waiter for 2 months as well to supplement his initial payments. Those 2 months ended up reducing the length of his payment period by about 8 months.*