# POLYNOMIALS: EXERCISES ON UNDERSTANDING

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## Contents



This is a collection of problems, sometimes with hints, aimed at illustrating concepts that we didn't cover in class. Some are easy, others are hard. I had promised some about Cardano's Formula, but there is a bit more here.

## 1. Exploring Polynomials

In this section, there are a collection of exercises that explore some of the ideas behind polynomials.

**Exercise 1.1.** When we multiply two polynomials together, we can use the method of detached coefficients. For example, if we wanted to multiply together the polynomials $x^3 + 3x^2 - 2x + 4$ and $2x^2 + x + 6$, we can do the following:

$$\begin{array}{rrrrrr}
 & & 1 & 3 & -2 & 4 \\
 & & & 2 & 1 & 6 \\
\hline
 & & 6 & 18 & -12 & 24 \\
 & 1 & 3 & -2 & 4 & \\
2 & 6 & -4 & 8 & & \\
\hline
2 & 7 & 5 & 24 & -8 & 24
\end{array}$$

Justify this algorithm, and learn how to read off the product of the two polynomials from it.

**Exercise 1.2.**
(1) Use the method described above to find the product of $4t^3 + 2t^2 + 7t + 1$ and $2t^2 + t + 6$.
(2) Evaluate each of these polynomials and their product at $t = 10$.
(3) Compare this multiplication with the pencil-and-paper long multiplication for the product of $4271 \cdot 216$.

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**Exercise 1.3.** Suppose $p(x)$ and $q(x)$ are two polynomials, and let $\deg(p(x))$ mean "the degree of the polynomial $p(x)$." What can you say about $\deg(p(x)+q(x))$? [*It's not always true that it's equal to the maximum of the two. When is this not the case?*] What can you say about $\deg(pq(x))$?

**Exercise 1.4.** How is $\deg((p \circ q)(x))$ related to $\deg((g \circ p)(x))$?

**Exercise 1.5.** Is it possible to find a polynomial, apart from the constant 0 polynomial, that is 0 everywhere? That is, can one find a polynomial $p(x)$ such that $p(x) = 0$ for all $x$? Try to justify your answer. [*This is not easy, although I bet you'll get the right answer. Remember how the degree and the number of roots are related?*].

**Exercise 1.6.** This is a cool exercise. Choose any quadratic polynomial $p(x) = ax^2 + bx + c$. Then compute $p(1) + p(4) + p(6) + p(7)$ on one hand, and $p(2) + p(3) + p(5) + p(8)$ on the other. What do you notice about the two sums? It turns out that this problem has some really interesting large generalizations. It is possible to find a split of the numbers from 1 to 16 such that this also works for any cubic.

## 2. Quadratics

**Exercise 2.1.** Remind yourself of the proof of the quadratic formula. In particular, we learned that if $p(x) = ax^2 + bx + c$, then by completing the square we can rewrite $p(x)$ as $p(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{1}{4a}(b^2 - 4ac)$. Try to organize this into the quadratic formula you know.

**Exercise 2.2.** The standard quadratic formula says that the roots of the polynomial $p(x) = ax^2 + bx + c$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The part under the radical, $D = b^2 - 4ac$, is called the **discriminant**. What happens if $D > 0$? If $D = 0$? If $D < 0$? Come up with an example of each case. [*In particular, note how many real solutions there are.*]

**Exercise 2.3.** If $a, b$ are two numbers, there are various types of thing we can mean when we mean 'average.' Often, we mean their arithmetic average: $\dfrac{a+b}{2}$. But there is another common type of average called a geometric average: $\sqrt{ab}$. If you've never seen this before, I recommend looking it up on wikipedia. In this question, we show that the arithmetic mean of two numbers is always at least as large as the geometric mean. Use the fact that both of the zeroes of the quadratic $(x - \sqrt{a})(x - \sqrt{b})$ are real, and knowledge of the discriminant from above, to show that $\dfrac{a+b}{2} \geq \sqrt{ab}$.

**Exercise 2.4.** This is significantly harder than the others so far. In this problem, we explain a derivation for one of the most important and powerful basic inequalities out there. Recall the summation notation: $\sum$.

(1) Suppose that $a_k, b_k$ are nonnegative real numbers. Then the function $p(t) = \sum_{k=1}^{n}(a_k t + b_k)^2$ is a quadratic polynomial in $t$. Explain why the discriminant is nonpositive.
(2) Use this to show the Cauchy-Schwarz Inequality (perhaps just do the case when $n = 2$, so that you only have $a_1, b_1, a_2, b_2$ to worry about):

$$\sum_{k=1}^{n} a_k b_k \leq \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$$

## 3. The Arithmetic of Complex Numbers

**Exercise 3.1.** Let $z = x + iy$, $w = u + iv$, $p = s + it$ be three complex numbers. Recall that the conjugate $\overline{z} = x - iy$. Show the following:

(1) $(zw)p = z(wp)$ [*If you just sort of do it, it will work out*]
(2) $zw = wz$
(3) $\overline{z + w} = \overline{z} + \overline{w}$
(4) $\overline{zw} = \overline{zw}$
(5) $\overline{z^2} = \overline{z}^2$ [*This could be done for free from the last bit, for example*]

**Exercise 3.2.** In class, we have mentioned the Fundamental Theorem of Algebra: every polynomial can be factored into a product of linear factors, although complex numbers might arise. Afterwards, I said (without proof or justification) that if $r = x + iy$ is a complex root of a polynomial with real coefficients, then $\overline{r}$ is also a root. That is, I said that roots appear in conjugate pairs. Let's see why this is: Suppose $p(t)$ is a polynomial with real coefficients. Show that for any complex number $z + a + bi$, $p(\overline{z}) = \overline{p(z)}$. Then reason that if $r$ is a complex zero of $p(t)$, then $\overline{r}$ is also a root by considering $\overline{p(r)}$.

## 4. Toward's Cardano's Formula

We now try to motivate Cardano's Formula a bit.

**Exercise 4.1.** Consider the cubic equation $p(x) = x^3 - 12x^2 + 29x - 18 = 0$. Our first goal is to get rid of the 'quadratic term.' It turns out that it's always possible to shift $x$ to the left or right to remove the quadratic term. Show here, for instance, that $p(t + 4)$ converts this equation to $t^3 - 19t - 30 = 0$.

**Exercise 4.2.** We now follow the general method of Cardano on $f(t) = t^3 - 19t - 30$.

(1) Set $t = u+v$ and get the equation $u^3+v^3+(3uv-19)(u+v)-30 = 0$.
(2) Argue that if we can find a $u, v$ such that $u^3 + v^3 - 30 = 0$ and such that $3uv = 19$, then the resulting $t = u + v$ will be a solution to our cubic.

(3) Solving for $v$ in the second equation and substituting into the first, show that $u^3$ is a root of the quadratic equation $x^2-19x+30^3/27=0$. Solve for $u$. [*There might be a few times where you might worry about taking either the positive or negative square root, etc. Unfortunately, not every combination works. But try a few times, and it should work out.*]
(4) Then solve for $v$, so that you find a root $u+v$.
(5) Completely factor $f(t)$. You can check your answer with the Rational Root test, as this actually has a rational root.
(6) Use this factorization to factor $p(x)$ from the last exercise.

**Exercise 4.3.** This is actually a collection of statements, not quite an exercise. If $f(x) = x^3 + px + q$, then Cardano's method will always find a solution. But just as the discriminant of a quadratic tells the character of the solutions, there is a cubic discriminant as well. It is $D = 27q^2 + 4p^3$. If $D > 0$, then the cubic has one real root and two imaginary roots. If $D = 0$, then the cubic has a double root. If $D < 0$, then the cubic has three distinct real roots. Discriminants are really useful.

**Exercise 4.4.** By means of a transformation, get rid of the quadratic term in the equation $x^3 - 15x^2 - 33x + 847$. Then verify that its discriminant is 0, so that there is a repeated root.

**Exercise 4.5.** The Quartic. This is a brief overview of Descartes' Method to solve the quartic. This is the same Descartes who unified algebra and geometry with the Cartesian plane, and the same Descarte as "Cogito ergo sum" - "I think, therefore I am."

(1) Any quartic can be rewritten to get rid of the cubic term.
(2) If $p(t) = t^4 + pt^2 + qt + r$, then $p(t)$ can always be written as the product of two factors: $p(t) = (t^2+ut+v)(t^2-ut+w)$ where $u, v, w$ satisfy the simultaneous system

$$v + w - u^2 = pu(w - v) \qquad\qquad = qvw = r$$

This is much harder to see, so we don't go into it.
(3) The trick here is to eliminate $v, w$ in the above system to get a cubic equation in $u^2$. Now that we know how to solve cubics, this leads to one root of the quartic. But again, this leaves a cubic, so one could conceivably solve the resulting cubic as well.

**Exercise 4.6.** This is another doable exercise. Sometimes, other (slick) methods can be used to solve things like the quartic or cubic. Some of these are interesting in their own right, and this is one of them. Consider the polynomial $p(x) = 2x^4 + 5x^3 + x^2 + 5x + 2 = 0$, and notice the symmetry of the coefficients (which is when this method works). Show that writing $x = t + \frac{1}{t}$ leads to the (easily-solved) equation $2t^2 + 5t - 3 = 0$. Solve this, and use the result to solve for the original $x$.

**Exercise 4.7.** To check the above work, show that the left side of $p(x)$ (above) can be written as the product of the two quadratic polynomials that arise in solving for $x$ once the two values of $t$ are known.

**Exercise 4.8.** Polynomials with the type of coefficient symmetry as in the last two exercises are known as **Reciprocal Polynomials**. That is to say that if $p(x) = ax^n+bx^{n-1}+cx^{n-2}+\cdots+cx^2+bx+a$, then $p(x)$ is **reciprocal**. Argue that if $r$ is a root of a reciprocal polynomial, then so is $\frac{1}{r}$ (which is how it got its name).

There we have it.