MIDTERM REVIEW EXERCISES

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Contents

1. Introduction

We have now covered three-fifths of the material from this course. While in many ways, this course has been cumulative and we have revisited much of the earlier material repeatedly, there are things that have been left out. In this packet, we will review some of the problem-types that we have come across and methods of finding their solution.

We will also take this time to combine our new skills, such as our knowledge of trigonometric identities and transcendental functions, with some of our old skills.

2. Elementary Functions

We started by reviewing basic factoring and graphing linear equations. We then worked on developing the algebraic and graphical qualities of polynomials. We learned to solve any quadratic equation that we will ever see (any cubic too, though that's much harder). Later came the Factor Theorem and the Rational Root Theorem, allowing us to solve more. This all culminated with rational functions. This first set of exercises covers this material.

2.1. Factoring and Solving Quadratics.

Example 2.1. One tool that we use with high frequency is factoring. We cannot get away from factoring, it turns out. Many tools revolve around factoring. We'll look at three major factoring tools here.

(1) $(x+y)^2 = x^2 + 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$
(2) $(x-y)(x+y) = x^2 - y^2$

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(3) Completing the square

Example 2.2. We can use $(x+y)^2 = x^2 = 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$ both ways. For example, when we see an expression that we'd like to expand, like $(\sin x + \cos x)^2$, we can immediately say that it is $\sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + 2\sin x \cos x$ without any FOILing. (further, if we are exceptionally clever, we might remember that $2\sin x \cos x = \sin(2x)$).

On the other had, we can factor functions quickly too. For example, when we are asked to find the roots of $e^{2x} - 4e^x + 4$, we recognize this as $(e^x - 2)^2$. Thus there is a double root at $x = \ln 2$.

Exercise 2.3. Expand the following without directly multiplying it out:

- $(3x+2y)^2$
- $(5x-4)^2$
- $(x+\cos x)^2$
- $(\sec x - \sin x)^2$
- $(e^x + e^{-x})^2$

Exercise 2.4. Factor the following:

- $\cos^2 x + 2\cos x \tan x + \tan^x x$
- $2e^{2x} + 6\sqrt{2}e^x + 9$
- $\sin^2 x + 8\sin x + 16$
- $2 + x^2 + x^{-2}$

Example 2.5. It is usually very easy to see cases where we have $(x-y)(x+y)$ and rewrite it as $x^2 - y^2$, but we sometimes need to approach it in the opposite direction. For example, if we want to find the roots of $\sin^2 x - 1/2$, we can do this quickly and easily with this factoring method. $\sin^2 x - 1/2 = (\sin x - 1/\sqrt{2})(\sin x + 1/\sqrt{2})$, and thus the solutions are $x = \pi/4 + n\pi/2$.

Exercise 2.6. Factor the following:

- $3x^2 - 2y^2$ (just because everything starts as 'squares' doesn't mean that they're the squares of pretty numbers)
- $4e^{2x} - 9$
- $2\tan x - 4$

Further, if we want to factor over the complex numbers, we might notice that $x^2 + y^2 = (x+iy)(x-iy)$. So we cam factor the following:

- $4x^2 + 9y^2$
- $2x^2 + 16y^2$

Example 2.7. There is a general form for completing the square that always works. If we have $x^2 + ax + b$, we can note that $x^2 + ax + \frac{a^2}{4} - \frac{a^2}{4} + b = (x+\frac{a}{2})^2 - \frac{a^2}{4} + b$. For example, $x^2+3x+5 = x^2+3x+\frac{9}{4}-\frac{9}{4}+5 = (x+\frac{3}{2})^2-\frac{9}{4}+5$. If we believe this pattern, we could skip the middle.

For example, if we had $x^2+10x+3$, we could write this as $(x+\frac{10}{2})^2-25+3$ without any of the middle expansions.

Exercise 2.8. Complete the square on the following:

- $x^2 + 18x + 2$
- $x^2 + 5x + 3$
- $\sin^2 x + 3\sin x + 2$
- $2x^2 + 3x + 5$ (the task here is to remember how to deal with the leading 2)

The goal is for these factoring techniques to feel like second nature. The less one needs to think about them, the better. The task of finding roots is largely equivalent to factoring, due to the Factor Theorem. In short, this says that if $p(x)$ is a polynomial and $p(r) = 0$, then $p(x) = (x-r)q(x)$ where $q(x)$ is a smaller degree polynomial. Thus we can pull out a linear factor of $(x - r)$.

Example 2.9. Anytime we see a quadratic, we should be happy. We can solve quadratics, always. Through factoring or the quadratic formula, we can always solve quadratics. In fact, we can solve them quickly. One shouldn't need to spend more than a minute on a quadratic in the form $ax^2+bx+c = 0$.

Exercise 2.10. Solve the following quadratics:

- $x^2 + 5x + 18 = 0$
- $4x^2 + 3x + 2 = 0$
- $2\sin^2 x + 3\sin x + 1 = 0$
- $3\cos^2 x + 8\sin x + 1 = 0$
- $2e^{4x} - 13e^{2x} + 1 = 0$

In cases (like above), remember that $\cos^2 x + \sin^2 x = 1$, and this can be modified to relate $\csc^2 x$ to $\cot^2 x$, or to relate $\sec^2 x$ to $\tan^2 x$. It's important to recognize 'hidden quadratics,' and to do the necessary work to transform a quadratic into a form that you can solve.

Exercise 2.11. Solve the following quadratics:

- $5\cot^2 x + 14\csc x + 1 = 0$
- $2\cos^2 x + 4\sin x + 2 = 0$
- $\tan^2 x + 5\sec x + 3 = 0$
- $(x - 2)^2 + 2x + 4 = (x - 1)$
- $(x - 3)^3 + (x - 1)(x + 1) = x^3 + 1$
- $\sqrt{x-2} + \sqrt{x+2} = 2$
- $\sqrt{x+1} + \sqrt{x-4} = \sqrt{x} + 5$

2.2. **Polynomial Inequalities.** Perhaps the best way of solving polynomial inequalities is to find its roots, make a sign chart, and just test on each side of each root. It is almost certainly the fastest. This is our general method:

Example 2.12. When we see a polynomial inequality $p(x) \geq 0$, we find the roots $r_1, \ldots, r_n$ of the polynomial. We then draw a number line, and see if the polynomial is positive or negative between each pair of roots. We use this to decide on our inequality. As an aside, I want to mention that this is one of the easiest types of questions to merit partial credit as long as you show your work, if you're in a graded situation.

Exercise 2.13. Will design and update soon.

2.3. **Rational Functions.**