MAPPING CLASS GROUPS

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1. Introduction

In this presentation, we give a brief introduction on mapping class groups of a surface and some of their properties. This paper is meant only as an introduction to the topic, and is intended to represent the material of a 45-minute presentation. This is a compilation of some of the introductory results from include Ivanov's survey [2], Farb and Margalit's freely distributed book [1], the notes from the *Master Class on Geometry* at Strasbourg in 2009 of Massuyeau [5], and Johnson's survey for the AMS [3].

2. Mapping Class Group Basics

We consider a compact, connected, orientable surface Σ. By the classification theorem of 2D closed manifolds, Σ is either a sphere, the connected sum of g tori, or the connected sum of k real projective planes. Here, we assume that we have a connected sum of $g \geq 1$ tori.

We define the *Mapping Class Group* of Σ to be the group of orientation-preserving homeomorphisms $\Sigma \to \Sigma$ whose restriction to $\partial\Sigma$ is the identity, up to isotopy among homeomorphisms of the same kind.

Definition 2.1. *The mapping class group of Σ is defined to be the group*

$$\mathcal{M}(\Sigma) := \mathrm{Homeo}^{+,\partial}(\Sigma)/\cong \tag{2.1}$$

Additional notations for the Mapping Class Group include $\mathrm{MCG}(\Sigma)$ and $\mathrm{Mod}(\Sigma)$. It should be noted that this definition is specific to the 2D manifold case. It is possible to extend this definition to higher dimensions, but we will not consider that here.

Remark 2.2. There is a topology known as the compact-open topology frequently defined on the set of continuous maps between topological spaces. Suppose we have two topological spaces X and Y and let $C(X,Y)$ denote the set of all continuous maps from X to Y. Then given a compact subset K of X and an open subset U of Y, let $V(K,U)$ denote the set of all $f \in C(X,Y)$ s.t. $f(K) \subset U$. Then the collection of such $V(K,U)$ is a subbase for the compact-open topology, i.e. the compact-open topology is the smallest topology containing the compact-open topology.

We can give the compact-open topology to the set $\mathrm{Homeo}^{+,\partial}(\Sigma)$. Then a continuous path $\rho : [0,1] \to \mathrm{Homeo}^{+,\partial}(\Sigma)$ is a continuous homotopy through homeomorphisms, and so is an isotopy between $\rho(0)$ and $\rho(1)$. As the mapping class group is a group of orientations up to isotopy, we have that $\mathcal{M}(\Sigma) = \pi_0\left(\mathrm{Homeo}^{+,\partial}(\Sigma)\right)$, the group of path-components of $\mathrm{Homeo}^{+,\partial}(\Sigma)$.

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Each element of $\mathcal{M}(\Sigma)$ is called a *mapping class*.

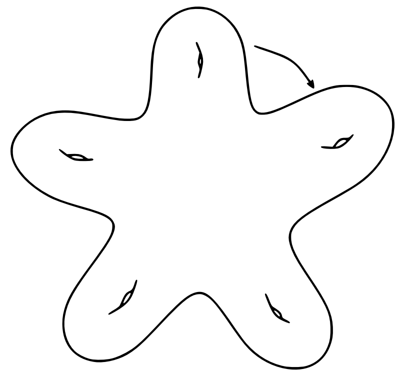

FIGURE 1. Σ_5 with an order 5 mapping class element indicated.

Let's find some examples of nontrivial mapping class elements. For our first example, we consider the surface of genus 5 indicated in Figure 1 above.

In the figure, we have indicated a homeomorphism ϕ of order 5, representing the homeomorphism that sends open sets on one 'wing' of the surface to the next 'wing,' and changes the center continuously in the expected way. This is perhaps non-intuitive, as this homeomorphism permutes the punctures. Such elements may permute the punctures, but they preserve boundary components pointwise.

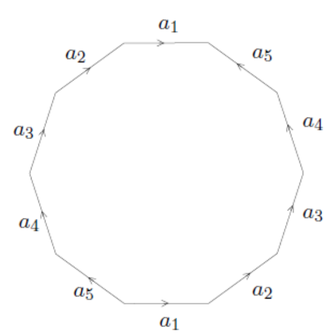

FIGURE 2. A representation of the genus 2 surface with opposite sides identified.

This clearly generalizes to cyclic mapping class elements of the mapping class group of Σ_g, the genus g torus. It is thus always easy to find an order g element on Σ_g. What about the other elements? If we look at figure 2, which represents Σ_g as a $(4g + 2)$-gon with opposite pairs of edges identified (similar to the cell structure we have used in class), then we note that rotating the $(4g + 2)$-gon any number of 'clicks' represents another mapping class. I use 'rotating by one click' to represent a $\frac{2\pi}{4g+2}$ radian rotation, of course.

In particular, a rotation by angle π is order 2 and gives an involution, called a 'hyperelliptic involution,' and is very important in the general theory of mapping class groups. In addition, this gives us many nontrivial elements. The 22-gon for Σ_5, for example, suggests there are mapping class elements of orders 2 (the involution) and 4 (the 3 click rotation). But what do either of these actually look like? It is very hard to conceptualize these mappings.

Although we found many finite order elements of the mapping class group, most elements have infinite order. The simplest such elements are Dehn twists, and we will study those in section 3. But first, we have a few more interesting examples to cover.

Proposition 2.3. *(Alexander's Lemma)*
The space $\mathrm{Homeo}^{+,\partial}(D^2)$ *is contractible. Correspondingly,* $\mathcal{M}(D^2) = \{1\}$*.*

Proof. View D^2 as a subset of $\mathbb{C}$ and let $|x|$ denote the modulus of $x \in \mathbb{C}$. For any homeomorphism $f : D^2 \to D^2$ that acts like the identity on the boundary, we define a homeomorphism $f_t : D^2 \to D^2$ for all $t \in [0,1]$ by

$$f_t(x) := \begin{cases} t \cdot f(x/t) & \text{if } 0 \leq |x| \leq t \\ x & \text{if } t \leq |x| \leq 1 \end{cases} \tag{2.2}$$

Then the map

$$H : \mathrm{Homeo}^{+,\partial}(D^2) \times [0,1] \to \mathrm{Homeo}^{+,\partial}(D^2), \ (f,t) \mapsto f_t \tag{2.3}$$

is an isotopy between the retraction of $\mathrm{Homeo}^{+,\partial}(D^2)$ to $\{\mathrm{Id}_{D^2}\}$ and the identity of $\mathrm{Homeo}^{+,\partial}(D^2)$. Thus, $\mathrm{Homeo}^{+,\partial}(D^2)$ deformation retracts to $\{\mathrm{Id}_{D^2}\}$, and is therefore contractible. In addition, $\mathrm{Homeo}^{+,\partial}(D^2)$ contracts, and so there is only one path component. Thus $\pi_0\left(\mathrm{Homeo}^{+,\partial}(D^2)\right) = \{1\}$. $\square$

Corollary 2.4. $\mathcal{M}(S^2) = \{1\}$

Proof. Let $f : S^2 \to S^2$ be an orientation-preserving homeomorphism, and let γ be a simple closed oriented curve in S^2. Then we must have that $f(\gamma)$ is isotopic to γ, and so we can assume that $f(\gamma) = \gamma$.

Now we use Alexander's Trick above to each of the two disks into which the curve splits in S^2, and so there is only one path component of $\mathrm{Homeo}^{+,\partial}$. $\square$

One can also show that the mapping class groups of the punctured disk is trivial (use Alexander's Lemma assuming the puncture is at the origin).

But the mapping class group of the torus $S^1 \times S^1$ is non-trivial.

Proposition 2.5. *Let* $a := [S^1 \times 1]$ *and* $b := [1 \times S^1]$*, so that* (a,b) *is the basis for* $H_1(S^1 \times S^1)$ over $\mathbb{Z}$*. Then the following map is a group isomorphism.*

$$\begin{aligned} M : \ \mathcal{M}(S^1 \times S^1) &\to \mathrm{SL}_2(\mathbb{Z}) \\ [f] &\mapsto f_* \end{aligned} \tag{2.4}$$

where the isotopy class $[f]$ *is sent to the matrix of the induced map* $f_* : H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$*, all homotopy groups interpreted over* $\mathbb{Z}$*.*

Proof. First, we show that M is a group homomorphism. That M is a homomorphism to $\mathrm{GL}_2(\mathbb{Z})$ is clear by the properties of matrix multiplication and this basis. We need to check that M goes only to matrices of determinent 1. Note that

$$\forall [f] \in \mathcal{M}(S^1 \times S^1), \quad M([f]) = \begin{pmatrix} f_*(a) \cdot b & f_*(b) \cdot b \\ -f_*(a) \cdot a & -f_*(b) \cdot a \end{pmatrix} \tag{2.5}$$

where we use $\cdot : H_1(S^1 \times S^1) \times H_1(S^1 \times S^1) \to \mathbb{Z}$ to mean the intersection numbers of the curves. This is the minimal number of intersections points between representative curves in the class a and a representative curve in the class b. We use this definition in greater detail at 3.1. While this can often be intractable, for simple cross products like this it is very simple and natural. Since f is orientation preserving, the intersection pairing is invariant. Nonetheless, we have then that

$$\det M([f]) = (f_*(b) \cdot b)(f_*(a) \cdot a) - (f_*(b) \cdot a)(f_*(a) \cdot b) = a \cdot b = 1 \tag{2.6}$$

So we have a group homomorphism to $\mathrm{SL}_2(\mathbb{Z})$.

To prove surjectivity, we realize $S^1 \times S^1$ as its natural cover $\mathbb{R}^2/\mathbb{Z}^2$ in the normal fashion, so that the loop $S^1 \times 1$ lifts to $[0,1] \times 0$ and $1 \times S^1$ lifts to $0 \times [0,1]$. Now any matrix $T \in \mathrm{SL}_2(\mathbb{Z})$ defines a linear homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ that that induces an orientation-preserving homeomorphism $t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. And $M([t]) = T$, so we have surjectivity.

To prove injectivity, consider a homeomorphism $f \ : S^1 \times S^1 \to S^1 \times S^1$ s.t. $M([f])$ is trivial. We know that $\pi_1(S^1 \times S^1)$ is abelian from our class work, and this implies that f acts trivially with respect to the fundamental group. We also know that $\mathbb{R}^2$ is the universal cover of $S^1 \times S^1$. So f can be lifted to a unique homeomorphism $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\tilde{f}(0) = 0$. By assumption on f, $\tilde{f}$ is $\mathbb{Z}^2$-equivariant. Therefore, we have a homotopy

$$H : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2, \quad (x,t) \mapsto t\tilde{f}(x) + (1-t)x \tag{2.7}$$

between $\mathrm{Id}_{\mathbb{R}^2}$ and $\tilde{f}$, and this homotopy is the lift of a homotopy from $\mathrm{Id}_{S^1 \times S^1}$ to f. For two dimensional manifolds, a homotopy is an isotopy, and so we see that $[f] = 1 \in \mathcal{M}(S^1 \times S^1)$. □

So we have used the universal cover to show that the mapping class group of the torus is infinite. This sort of technique of using the universal cover to compute the mapping class group works at least partially for many surfaces. The mapping class group for the annulus can be computed in the exact same way. (It happens to be isomorphic to $\mathbb{Z}$.

3. Dehn Twists

But that technique is not the best way to examine mapping class groups. We will now consider certain homeomorphisms called 'Dehn twists.' These are homeomorphisms $\Sigma \to \Sigma$ whose support is the regular neighborhood of a simple closed curve. Often, we can take these curves to be simple circles or other toy shapes.

We will again use the intersection number.

Definition 3.1. *Given two simple closed curves α and β on Σ, we define their* geometric intersection number *by*

$$i(\alpha, \beta) := \min\{|\alpha' \cap \beta'| \alpha' \text{ isotopic to } \alpha, \beta' \text{ isotopic to } \beta\} \tag{3.1}$$

Definition 3.2. *Let α be a simple closed curve on Σ. We choose a closed regular neighborhood N of α in Σ and we identify it with $S^1 \times [0,1]$ in such a way that orientations are preserved. Then the* Dehn Twist *along α is the homeomorphism $\tau_\alpha \ : \Sigma \to \Sigma$ defined by*

$$\tau_\alpha(x) = \begin{cases} x & \text{if } x \notin N \\ (e^{2i\pi(\theta+r)}, r) & \text{if } x = (e^{2i\pi\theta}, r) \in N = S^1 \times [0,1] \end{cases} \tag{3.2}$$

And the isotopy class of τ_α depends only on the isotopy class of the curve α.

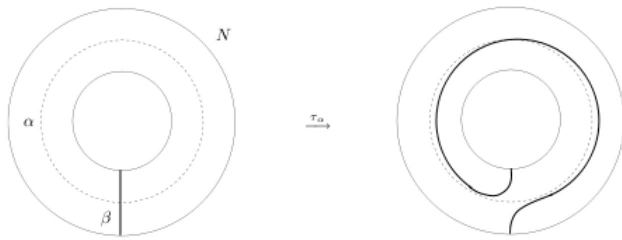

FIGURE 3. The effect of τ_α on a transversal curve β

In the figure above, Figure 3, we show the effect of a Dehn twist on a transversal path. This picture illustrates that this local change does not affect global connectivity. This leads to the following observations about Dehn twists: firstly, τ_α has infinite order whenever $[\alpha] \neq 1 \in \pi_1(\Sigma)$; secondly, the conjugate of a Dehn twist is again a Dehn twist. In fact, if $f : \Sigma \to \Sigma$ is an orientation-preserving homeomorphism, then we have that

$$f \circ \tau_\alpha \circ f^{-1} = \tau_{f(\alpha)} \tag{3.3}$$

We see this quickly, as outside of the regular neighborhood N associated to τ_α, the homeomorphism f and its inverse f^{-1} cancel. And within the neighborhood, we see the resulting homeomorphism is generated by the modified image of α by f.

We can immediately use Dehn twists to investigate the mapping class group of the annulus. Using the 'center circle' of the annulus and choosing a relatively small cylinder (less then half the difference of the radii of the annulus), we have an infinite cyclic subgroup. It turns out that $\mathcal{M}(S^1 \times [0,1])$ is infinite cyclic generated by τ_α, and so is the entire group. This is the motivation for the following theorem, one of the two primary theorems of this presentation.

Theorem 3.3. *(Dehn)*
The group $\mathcal{M}(\Sigma)$ is generated by Dehn twists along circles which are non-separating, or which encircle some boundary components.

Before we prove this result, we need the following theorem.

Theorem 3.4. *(Birman's exact sequence)*
Let Σ' be the compact oriented surface obtained from Σ by removing a disk D. Then there is an exact sequence of groups

$$\pi_1(U(\Sigma)) \to \mathcal{M}(\Sigma') \to \mathcal{M}(\Sigma) \to 1 \tag{3.4}$$

where $U(\Sigma)$ denotes the total space of the unit tangent bundle of Σ. Moreover, the image of the 'Push map', the map $\pi_1(U(\Sigma)) \to \mathcal{M}(\Sigma')$, is generaged by some products of Dehn twists along curves which are non-separating or which encircle boundary components.

We will not cover the proof behind this exact sequence here. It simply takes too long and we have too much to do. But the freely available Farb and Margalit [1] contains a complete proof.

We will also use the following result from Lickorish [4] without proof.

Proposition 3.5. *(Connectedness of the complex of curves)*

On surfaces Σ *of genus* $g \geq 2$*, for any two separating curves* α *and* α'*, there exists a sequence of non-separating curves*

$$\alpha = \alpha_1, \alpha_2, ..., \alpha_r = \alpha' \tag{3.5}$$

such that $i(\alpha_i, \alpha_{i+1} = 0 \; \forall \; j = 1, \cdots, r-1$

These two results let us complete the proof of Theorem 3.3.

Proof. We proceed by induction on the genus g. Before we continue, we consider a technical issue (in fact, this is why we need the Birman exact sequence). Although we have only been considering closed surfaces with no boundary components, we require boundary components for this proof. But only to apply the Birman exact sequence. In fact, by considering Theorem 3.4, we can say that if this proof holds for a given genus g and 0 boundary components, then it holds for genus g and any number of boundary components (by repeatedly removing disks and looking at the result). So in particular, we appeal to the zero boundary component case for each genus, and all the non-zero boundary component cases follow. For $g = 0$, we have already proved in Corollary 2.4 that $\mathcal{M}(S^2)$ is trivial. It is thus trivially generated by the Dehn twists. For $g = 1$, we have shown in Proposition 2.5 that $\mathcal{M}(S^1 \times S^1)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. It is well-known that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{3.6}$$

These correspond to the Dehn twists along the meridional curve (say $[1 \times S^1]$) and the equatorial curve ($[S^1 \times 1]$).

So now, assume that the genus g is at least 2. Let $f \in \mathcal{M}(\Sigma)$ and suppose α is a non-separating simple closed curve on Σ. Then $f(\alpha)$ is another non-separating simple closed curve on Σ as well. Note that we can now appeal to Proposition 3.5. Note that we can interpret Proposition 3.5 with Dehn twists as follows:

Corollary 3.6. *Let* Σ *be a surface of genus* $g \geq 2$*. Then for any two non-separating simple closed curves* β *and* γ *on* Σ *with* $i(\beta, \gamma) = 0$*, there is a product of Dehn twists* τ *along non-separating simple closed curves s.t.* $\tau(\beta) = \gamma$*.*

In fact, if β and γ are non-separating simple closed curves in Σ, then τ_β and τ_γ are conjugate in $\mathcal{M}(\Sigma)$. Back to our proof, we can find a product of Dehn twists T along non-separating simple closed curves such that $T(\alpha) = f(\alpha)$. So we can assume that f preserves α. (There is the case that f inverses the orientation of α, in which case we consider a non-separating simple closed curve β such that $\imath(\alpha, \beta) = 1$. Then $\tau_\beta \tau_\alpha^2 \tau_\beta$ preserves α but inverses its orientation)

Further, we understand the orientation-preserving homeomorphisms of S^1 completely, and in fact there is exactly one such homeomorphism up to isotopy. So we may assume that f is the identity on α and accordingly on a neighborhood M of α. There is a closed neighborhood N such that $\alpha \subset N \subset M$.

Let $\Sigma' := \Sigma \setminus N^\circ$, where N° denotes the interior of N. Then Σ' has genus $g - 1$ (and two more boundary components). Then, since a non-separating simple closed curve in Σ' is non-separating in Σ, and a boundary curve in Σ' is isotopic to α (also non-separating), we conclude by the induction hypothesis and our technical detail from the Birman exact sequence.

□

The proof given simply shows existence. It is not at all obvious that a finite number of generators generates these groups. But the above proof can be improved. In fact, for the closed surface Σ_g of genus $g \geq 1$, Lickorish ([4]) showed the following theorem:

Theorem 3.7. *(Lickorish)*
For $g \geq 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along the $3g-1$ circles indicated in Figure 4 below.

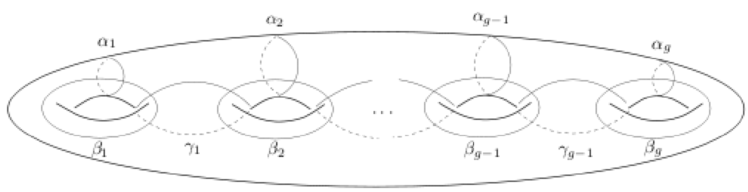

FIGURE 4. the Lickorish generators

4. Concluding Remarks

I hope that this brief presentation served to illustrate some of the preliminary results on mapping class groups and mapping classes themselves. Like other algebraic invariants of topological spaces, mapping class groups play into many different areas of math. To conclude, I wanted to mention two aspects of mapping class groups that I did not mention in this talk.

Firstly, some mapping class groups are better studied than others. The mapping class groups of the punctured disks are called the braid groups, and in a certain sense they are generalizations of the symmetric group. It's called a 1braid group' because it refers to the number of ways to 'braid' a certain number of strings (up to a few restrictions - one doesn't cut any strings and strings don't ever go backwards - it's like a life guard lanyard). If one associates each puncture on a punctured disk with a line from that puncture to the boundary of the disk, then the mapping class homeomorphisms permute these punctures and their lines. These lines are like 'strings' (like those that form a braid) and the resulting braid is the element of the braid group. It is a generalization of the symmetric group because if one ignores the details of the braid itself, the braid group is just a permutation on the strings themselves (with respect to the punctured disk, this corresponds to the mapping class homeomorphisms that permute the punctures).

Second, the mapping class group induces an action on the homology and cohomology of the space. The kernel of this action is called the Torelli group, perhaps the most studied group associated to mapping class groups. But that's another talk, another day.

References

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