A METHOD TO CALCULATE THE NUMBER OF LATTICE POINTS BELOW A QUADRATIC

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ABSTRACT. In this, I present a method of quickly counting the number of lattice points below a quadratic of the form $y = \frac{a}{q}x^2$. In particular, I show that knowing the number of lattice points in the interval $[0, q-1]$, then we have a closed form for the number of lattice points in any interval $[hq, (h+1)q-1]$. This method was inspired by the collaborative Polymath4 Finding Primes Project [1], and in particular the guidance of Dr. Croot from Georgia Tech.

1. Intro

Suppose we have the quadratic $f(x) = \frac{a}{q}x^2$. In short, we separate the lattice points into regions and find a relationship between the number of lattice points in one region with the number of lattice points in other regions. Unfortunately, the width of each region is $q$, so that this does not always guarantee much time-savings.

This came up while considering

$$\sum_{d \leq x \leq m} \left\lfloor \frac{N}{x} \right\rfloor$$

In particular, suppose we write $x = d + n$, so that we have $\left\lfloor \frac{N}{d + n} \right\rfloor$. Then, expanding $\frac{N}{d + n}$ like $\frac{1}{x}$, we see that

$$\frac{N}{d + n} = \frac{N}{d} - \frac{N}{d^2} (n - d) + O \left( \frac{N}{d^3} \cdot (n - d)^2 \right)$$

And correspondingly, we have that

$$\sum \left\lfloor \frac{N}{d + n} \right\rfloor = \sum \left[ \frac{N}{d} - \frac{N}{d^2} (n - d) + O \left( \frac{N}{d^3} \cdot (n - d)^2 \right) \right]$$

Now, I make a great, largely unfounded leap. This is almost like a quadratic, so what if it were? And then, what if that quadratic were tremendously simple, with no constant nor linear term, and with the only remaining term having a rational coefficient? Then what could we do?

2. The Method

We want to find the number of lattice points under the quadratic $y = \frac{a}{q}x^2$ in some interval. First, note that

$$\left\lfloor \frac{a}{q} (x + q)^2 \right\rfloor = \left\lfloor \frac{a}{q} (x^2 + 2xq + q^2) \right\rfloor = \left\lfloor \frac{a}{q} x^2 \right\rfloor + 2ax + aq$$
Then we can sum over an interval of length $q$, and we’ll get a relationship with the next interval of length $q$. In particular, this means that

$$\sum_{x=0}^{q-1} \left\lfloor \frac{a x^2}{q} \right\rfloor = \sum_{x=q}^{2q-1} \left\lfloor \frac{a x^2}{q} \right\rfloor - \sum_{x=0}^{q-1} (2ax + aq)$$

Now I adopt the notation $S_{a,b} := \sum_{x=a}^{b} \left\lfloor \frac{a x^2}{q} \right\rfloor$, so that we can rewrite equation (5) as

$$S_{0,q-1} = S_{q,2q-1} - \sum_{0}^{q-1} (2ax + aq)$$

Of course, we quickly see that we can write the right sum in closed form. So we get

$$S_{0,q-1} = S_{q,2q-1} - a(q - 1)(q) - aq^2$$

We can extend this by noting that $\frac{a}{q}(x +hq)^2 = \frac{a}{q}x^2 + 2ahx + ahq$, so that

$$S_{0,q-1} = S_{hq,(h+1)q-1} - \sum_{0}^{q-1} (2ahx + ahq)$$

Extending to multiple intervals at once, we get

$$\lambda S_{0,q-1} = \sum_{h=1}^{\lambda} \left( S_{hq,(h+1)q-1} - h \sum_{0}^{q-1} (2ax + aq) \right) = S_{q,\lambda+1q-1} - \sum_{h=1}^{\lambda} h \left( \sum_{0}^{q-1} (2ax + aq) \right) = S_{q,\lambda+1q-1} - \frac{\lambda(\lambda + 1)}{2} [aq(q + 1) + aq^2]$$

So, in short, if we know the number of lattice points under the parabola on the interval $[0, q-1]$, then we know in $O(1)$ time the number of lattice points under the parabola on an interval $[0, (\lambda + 1)q - 1]$. Unfortunately, when I have tried to take this method back to the Polymath4-type problem, I haven’t yet been able to reign in the error terms. But I suspect that there is more to be done using this method.

References