Zeros of half integral weight dirichlet series

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Acknowledgements

This project began at the Nesin Mathematical Village in Turkey, and wouldn't have started without the initial organizational effort of Mehmet Kıral (in blue on the right).

This reflects a collaboration with Mehmet Kıral, Li-Mei Lim, and Thomas Hulse. (But all mistakes are probably my own).

The Selberg Class

Typical L-functions

Half-integral weight modular forms

Computational Results

The Selberg Class of L-functions

We frequently study the Selberg Class of L-functions. These are Dirichlet series

$$L(s) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

that satisfy

1. **Analytic Continuation**: $L(s)$ has analytic continuation to $\mathbb{C}$ (with the possible exception of a pole at $s = 1$).
2. **Ramanujan Conjecture**: The coefficients grow slowly, $|a(n)| \ll n^{\epsilon}$ for any $\epsilon > 0$.
3. **Functional Equation**: $L(s)$ can be completed $\Lambda(s) = L(s) Q^s G(s)$ for a (real) number Q and a product of Gamma factors $G(s)$, such that $\Lambda(s) = \epsilon \overline{\Lambda(1 - \overline{s})}$ for some $|\epsilon| = 1$.
4. **Euler Product**: $L(s)$ has an Euler product $L(s) = \prod_p L_p(s)$ for "nice" objects $L_p(s)$.

It turns out that to many objects of arithmetic interest, we can associate an L-function that is (perhaps conjecturally) in the Selberg Class. And all of these L-functions are conjectured to satisfy similar properties to the $\zeta(s)$.

For example, a Generalized Riemann Hypothesis is conjectured for Selberg Class L-functions: all (nontrivial) zeros of $L(s)$ should be on the line $\operatorname{Re} s = \frac{1}{2}$.

Modular L-functions

One source of L-functions are modular forms. A (weight k, holomorphic) modular form is a holomorphic function f on the upper half-plane $\mathcal{H} = \{x + iy : (x, y) \in \mathbb{R}^2, y > 0\}$, which transforms in a prescribed way under the action of a matrix group $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$:

$$f(\gamma z) := f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}(2, \mathbb{Z}).$$

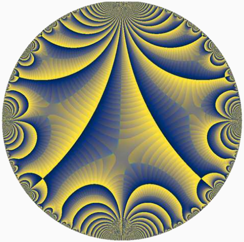
Further, we require $f(z)$ to be holomorphic on the compactified quotient $\Gamma \backslash \mathcal{H}$

To each normalized modular (and cuspidal, Hecke) eigenform f, we can associate an L-function

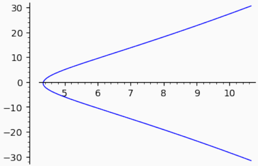
$$L(s, f) = \sum_{n \geq 1} \frac{a(n)}{n^{s + \frac{k-1}{2}}} = \prod_p \left(1 - \alpha_p p^{-s}\right)^{-1} \left(1 - \beta_p p^{-s}\right)^{-1}.$$

The Delta function

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24} \quad (q = e^{2\pi iz})$$
$$= q - 24q^2 + 252q^3 + \cdots$$



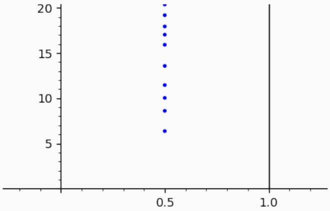
is a weight 12 modular form on $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. The L-function satisfies the functional equation $\Lambda(s,f) := (2\pi)^{-s}\Gamma(s+\frac{11}{2})L(s,f) = \Lambda(1-s,f)$.



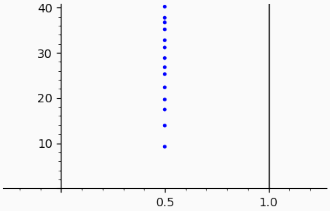
The Elliptic curve $Y^2 + Y = X^3 - X^2 - 10X - 20$. Let $a(p) = (p+1) - \#E(\mathbb{F}_p)$ count deviation from the expected number of solutions on the curve over $\mathbb{F}_p$.

Then there is a weight 2 modular form on $\Gamma(11) \subset \mathrm{SL}(2,\mathbb{Z})$, whose coefficients $a(p)$ match exactly the $a(p)$ defined above, and the L-functions associated to the curve and this modular form are the same and satisfy $\Lambda(s,E) := (22\pi)^{s/2}\Gamma(s+1/2)L(s,E) = \Lambda(1-s,E)$.

Both of these examples give Selberg L-functions. The analyticity and functional equation follow from the action of the matrix subgroup of $\mathrm{SL}(2,\mathbb{Z})$ on the modular form. The Euler product comes from the theory of Hecke operators. The Ramanujan conjecture $|a(n)| \ll n^\epsilon$ is the Hasse-Weil Bound on elliptic curves (for $L(s,E)$) or the highly nontrivial Deligne's Bound [Del71] (for general modular $L(s,f)$).



First zeros of $L(s,E)$



First zeros of $L(s,\Delta)$

A similar, but different, story

Typical L-functions

Half-integral weight modular forms

Computational Results

No Euler products

Of the four requirements of the Selberg Class (analyticity, Ramanujan conjecture, a functional equation, and an Euler product), the most surprising to me is the Euler product. The other requirements all feel very "analytic", but the Euler product is feels fundamentally "arithmetic".

But it's known that a (nice) Euler product is essential to results like RH.

Davenport–Heilbronn series

For example, Davenport and Heilbronn [DH36] studied a Dirichlet series formed from a particular linear combination of Dirichlet L-functions,

$$L(s) = \frac{1 - i\theta}{2} L(s, \chi) + \frac{1 + i\theta}{2} L(s, \overline{\chi}),$$

where θ is a particular constant and $\chi = \chi_5(2, \cdot)$ is the unique primitive character mod 5 with $\chi(2) = i$. Then $L(s)$ satisfies the functional equation

$$\Lambda(s) := L(s)\Gamma(\tfrac{s+1}{2})(5/\pi)^{s/2} = \Lambda(1 - s),$$

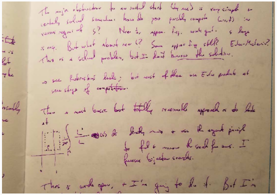
but has infinitely many zeros on the critical line and infinitely many zeros in the half-plane $\operatorname{Re} s > 1$.

The exceptional zeros appear to be sporadic: there are four zeros off the critical line with $0 < \operatorname{Im} s < 200$. Nonetheless, one can show there are infinitely many.

My Original Motivation

More generally, we expect that any Dirichlet series that satisfies the first three requirements of the Selberg class *but not an Euler product* should fail to satisfy a Riemann Hypothesis.

Until 2018, the only sort of example of this sort of not-quite-Selberg Dirichlet series and analysis I'd seen were formed from linear combinations of Selberg Class L-functions, like the Davenport–Heilbronn series.



But there is a class of Dirichlet series coming from half-integral weight modular forms, which (we think) aren't linear combinations of Selberg Class L-functions, and which don't have a multiplicative structure.

Half-integral weight modular forms

A modular form of half-integral weight k (so k here is in $\frac{1}{2} + \mathbb{Z}$) is a holomorphic function on $\mathcal{H}$ that transforms in a prescribed way under the action of a discrete subgroup $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$, satisfying

$$f(\gamma z) = j(\gamma, z)^k f(z)$$

for a half-integral factor of automorphy $j(\gamma, z)$. In the remainder of this talk, I'll consider the cocycle

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{cz + d}, \qquad \varepsilon_d = \begin{cases} 1 & d \equiv 1 \bmod 4, \\ i & d \equiv 3 \bmod 4. \end{cases}$$

As with full-integer weight forms, we require that f be *holomorphic at all the cusps* and to have a Fourier expansion

$$f(z) = \sum_{n \geq 0} a(n) e(nz).$$

Dirichlet series

Half-integral weight cusp forms of weight k on a matrix group Γ form a complex vector space $S_k(\Gamma)$. To any such cuspform $f(z) = \sum_{n \geq 1} a(n)e(nz)$, one can associate a Dirichlet series

$$L(s, f) = \sum_{n \geq 1} \frac{a(n)}{n^{s + \frac{k-1}{2}}},$$

but these Dirichlet series won't have Euler products, even if f is a Hecke eigenform.

Each such Dirichlet series have analytic continuation to $\mathbb{C}$ and satisfy a functional equation of the form

$$Q^s L(s, f) G(s) = \epsilon Q^{1-s} L(1 - s, g) G(1 - s),$$

where g is a modular form related to f via an involution of the form $g(z) \approx f(1/Nz)$. But in general (in contrast to the full-integral case), g is not a cusp form in the same space $S_k(\Gamma)$ — in general g can transform with a quadratically twisted factor of automorphy $\chi_N(\gamma) j(\gamma, z)^k$.

A priori, there are thus two differences between Dirichlet series coming from half-integral weight modular forms and the Selberg class: a typical half-integral weight modular form doesn't yield a symmetric functional equation, *and* the Dirichlet series won't have an Euler product.

However, for Hecke eigenforms on $\Gamma_0(4N)$, for N squarefree, it is possible to choose a related form with a symmetric functional equation.

Lemma

Let $f(z)$ be a Hecke eigenform of half-integral weight k on $\Gamma_0(4N)$ with (full-integer) weight $2k-1$ Shimura correspondent F. Then there is Hecke eigenform g of weight k on $\Gamma_0(16N^2)$ that also has Shimura correspondent F and whose Dirichlet series satisfies the symmetrical functional equation

$$\Lambda(s,g) = Q^s L(s,g) G(s) = \epsilon \Lambda(1-s,g)$$

for some $|\epsilon| = 1$.

(Aside: Frequently one can take g to be on $\Gamma_0(4N^2)$).

We now consider only those symmetrized half-integral weight forms. Each such form has a Dirichlet series $L(s,g)=\sum_{n\geq 1} A(n)n^{-s}$ that has analytic continuation to $\mathbb{C}$ and a symmetric functional equation (Selberg class requirements 1 and 3). Further, one can show

$$\sum_{n\leq X} |A(n)|^2 \sim c_g X,$$

so that the Ramanujan Conjecture $A(n) \ll n^{\epsilon}$ is true on average.

Such a Dirichlet series $L(s,g)$ is *very similar* to a Selberg Class L-function like $\zeta(s)$. Classical proofs that *completely avoid the Euler product* and that *don't expect the logarithmic derivative L'/L to behave* carry through.

For example, one can prove the following results: three are *typical* (in the Selberg class sense) and two are a bit *atypical* (in that they're trivial or vacuous for the Selberg class).

Theorem

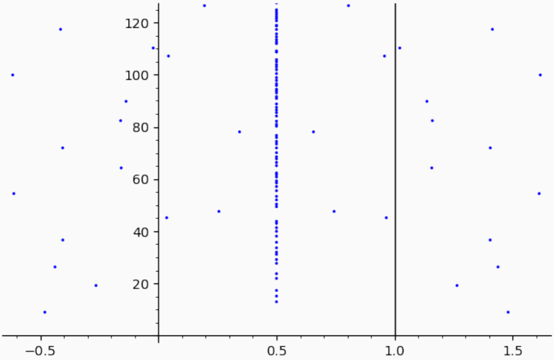
- *$L(s, g)$ has on the order of $T \log T$ nontrivial zeros with $0 < \operatorname{Im} s < T$.*
- *$L(s, g)$ has at most $\log T$ zeros (counting multiplicity) in any strip $T < \operatorname{Im} s < T + 1$.*
- *For any $\epsilon > 0$, almost all (i.e. 100 percent of the) zeros of $L(s, g)$ occur within ϵ of the critical line.*
- *All nontrivial zeros of $L(s, g)$ are constrained to a strip $1 - A < \operatorname{Re} s < A$. (But typically $A > 1$).*
- *If $L(s, g)$ has at least one zero in the region $\operatorname{Re} s > 1$, then $L(s, g)$ has infinitely many, and there are $\Omega(T)$ in the region $0 < \operatorname{Im} s < T$.*

Computational Results

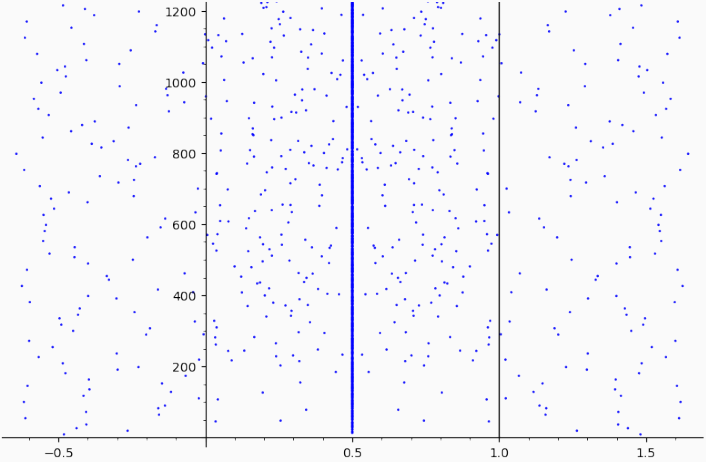
Typical L-functions

Half-integral weight modular forms

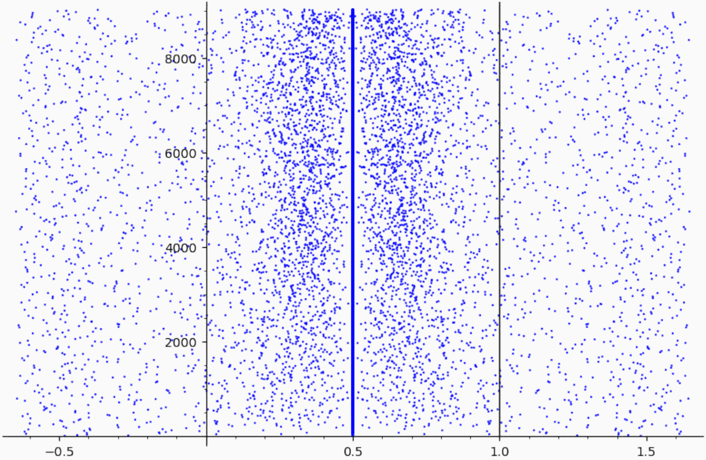
Computational Results



These are zeros of the unique half-integral weight modular form g of weight $9/2$ on $\Gamma_0(4)$ (a form appearing in Shimura's paper [Shi73]). If $\eta(z) = e(z/24)\prod_{n\geq 1}(1 - e(nz))$ is the Dedekind η function (a 24th root of $\Delta(z)$), then this form is $g(z) = \eta(2z)^{12}\theta(z)^{-3}$.



Yoshida [Yos95] computed the first couple dozen of these zeros in 1995.



About 70 percent of the zeros in this image are on the critical line.

To find the zeros, we use a triple of techniques.

1. Zeros on the critical line
2. Quick heuristic methods for zeros off the critical line
3. Verification and checking for zeros off the critical line

Finding zeros on the critical line can be done pretty quickly with an analog of the Hardy Z function. There is a sign ϵ_1 such that $\epsilon_1 \Lambda(s, g)$ is real valued on the critical line. Then one computes $\Lambda(s, g)$ and looks for sign changes.

Theorem

For each symmetrized L-function $L(s, g)$, there are infinitely many zeros on the critical line.

Zeros off the critical line

To find zeros off the critical line, we've turned towards using Newton's Method of root finding (which works very well when it works, as $L(s, g)$ is complex analytic and all roots we've found are single roots).

That is, we compute several iterations of the map

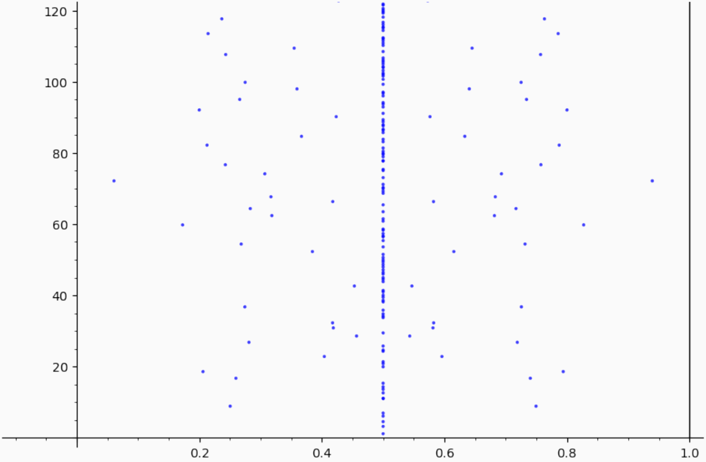
$$s_n = s_{n-1} - \frac{L(s_{n-1}, g)}{L'(s_{n-1}, g)}$$

on a mesh of points. We ignore iterations that diverge and collect the various remaining candidate zeros for later verification.

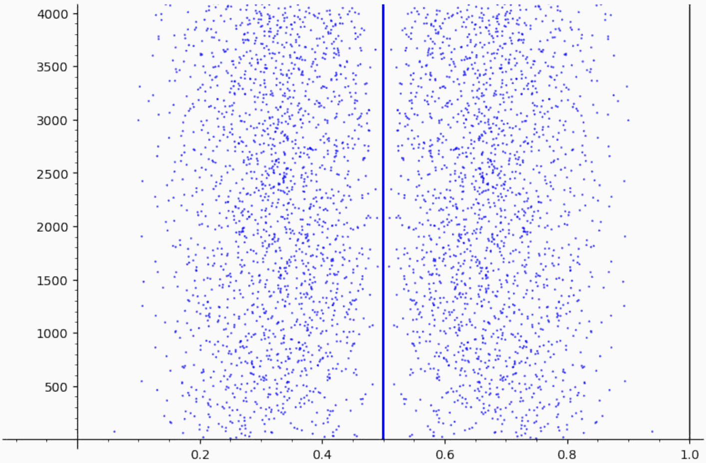
To verify counts and locations of zeros, we numerically compute integrals

$$\frac{1}{2\pi i} \int_C \frac{L'(z, g)}{L(z, g)} dz$$

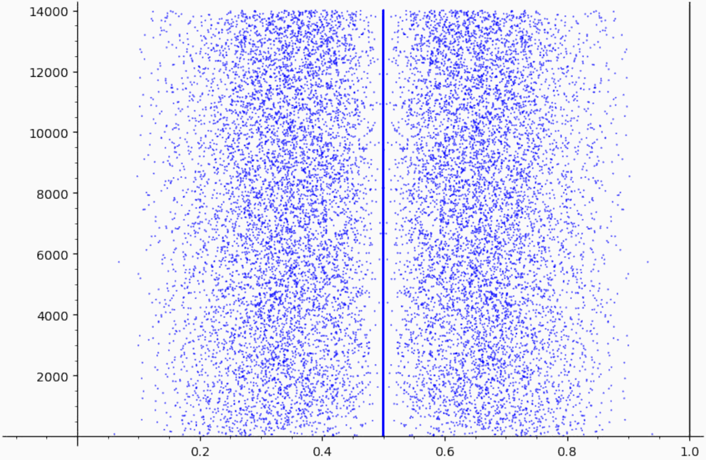
over contours C around heuristic zeros. By the argument principle, this integral gives the number of zeros (with multiplicity) inside the contour.



A weight 9/2 form on $\Gamma_0(12)$.



There don't seem to be any zeros outside of the critical strip. (I don't know how to show that these do or don't occur).



Approximately 65 percent of these zeros are on the critical line.

Pair Correlation

Typical L-functions

Half-integral weight modular forms

Computational Results

We can investigate statistics concerning the zeros as well. One nontrivial statistic is the pair correlation. For Selberg L-functions, the pair correlation is defined in terms of the spacing between the nontrivial zeros, weighted so that the expected spacing is 1 on average. Here, as we have lots of "exceptional" zeros, it's not clear what the right analogue is.

We investigated the pair correlation between the imaginary parts of zeros, normalized so that the typical spacing is 1 on average. That is, if $\rho_n = \sigma_n + i\gamma_n$ is the nth zero, then we consider spacings

$$\delta_n = c(\gamma_{n+1} - \gamma_n) \log c' \gamma_n,$$

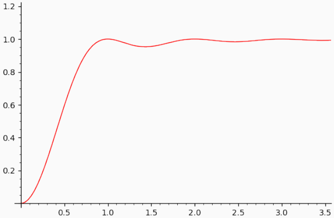
where c and c' come from the zero count $N(T)$ of zeros up to height T.

Then the pair correlation function is the distribution $\phi(u)$ such that as $M, N \to \infty$,

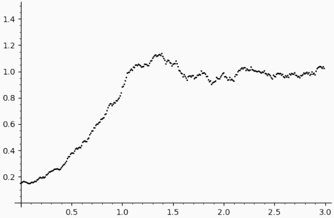
$$\frac{1}{M} \Big\{ (n, k) : N \leq n \leq N+M, k \geq 0, \delta_n + \cdots + \delta_{n+k} \in [\alpha, \beta] \Big\} \sim \int_\alpha^\beta \phi(t) dt,$$

(if this function exists).

The Montgomery Pair Correlation Conjecture posits that the pair correlation function for $\zeta(s)$ is $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$.

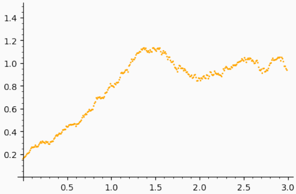


Computationally, the pair correlation function for the (normalized differences between imaginary parts of the) zeros of the weight 9/2 form on $\Gamma_0(4)$ look like the figure at right.

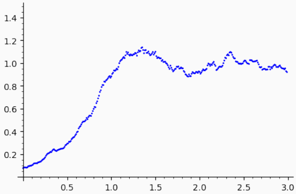


Qualitatively, these look similar. There is a similar repulsion phenomenon initially, and the shape is roughly similar. But they're also clearly not the same.

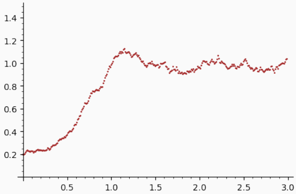
Let's examine estimated pair correlation functions for other forms.



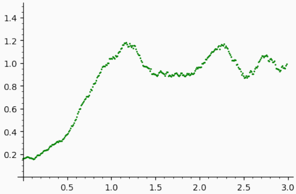
75k zeros of a weight 13/2 form on $\Gamma_0(4)$



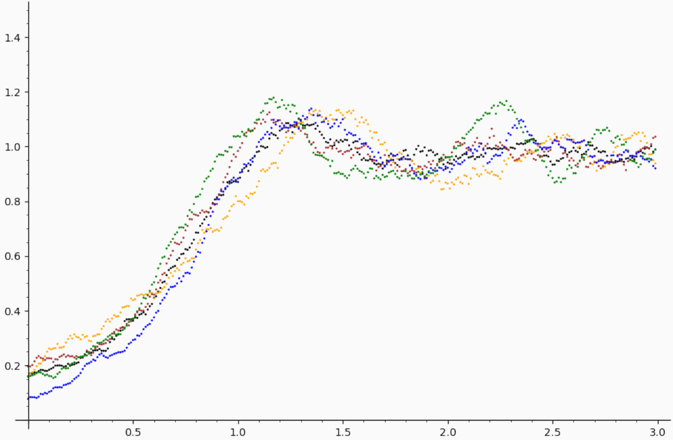
40k zeros of a weight 9/2 form on $\Gamma_0(12)$



70k zeros of a weight 15/2 form on $\Gamma_0(4)$



10k zeros of another weight 9/2 form on $\Gamma_0(12)$

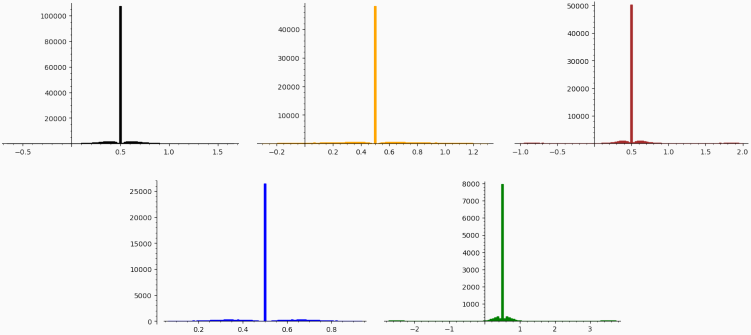


All five computed pair correlation approximations, plotted together. Notice how structurally similar they are, despite coming from different modular forms and over different ranges of zeros.

We don't know how to explain this.

We don't quite know where we're going, but what started as a purely exploratory investigation into a niche between linear combinations of L-functions and the Selberg class has transformed into an interesting little chestnut.

To end, I'll put (color coded) histograms of the real parts of the zeros we've computed for these five forms.



Thank you very much.

Please note that these slides (and references for the cited works) are (or will soon be) available on my website (davidlowryduda.com).

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