Zeroes of L-functions associated to half-integral weight modular forms

Joint Math Meetings

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Meeting the Key Figures

Full and Half Integral Weight Modular Forms

A modular form of weight k (so that k may be an integer or a half-integer in this talk) associated to a discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function f satisfying

$$f(\gamma z) = j(\gamma, z)^k f(z), \qquad \gamma \in \Gamma,$$

and some growth conditions concerning limiting behavior of f.

Here, $j(\gamma, z)$ is the factor of automorphy. The classical, full-integral weight factor is

$$j(\gamma, z) = (cz + d), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

When k is a half-integer, some attention is necessary in the definition since square-roots are around. The half-integral factor is essentially

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{cz + d}, \qquad \varepsilon_d = \begin{cases} 1 & d \equiv 1 \bmod 4 \\ i & d \equiv 3 \bmod 4 \end{cases}.$$

L-series associated to cusp forms

Each modular form f has a Fourier expansion

$$f(z) = \sum_{n \geq 0} a(n) e(nz).$$

Those forms with $a(0) = 0$ (with corresponding behavior at all the cusps) are called cusp forms.

To each cusp form, we can associate an L-series

$$L(s, f) = \sum_{n \geq 1} \frac{a(n)}{n^{s+(k-1)/2}},$$

which satisfies several remarkable properties.

1. $L(s,f)$ has analytic continuation to $\mathbb{C}$, and converges absolutely for $\operatorname{Re} s > 1$,
2. $L(s,f)$ satisfies a functional equation roughly of the form

$$L(s,f)G(s) \approx L(1-s,f)G(1-s),$$

   where G is some product of Γ functions,
3. $L(s,f)$ contains arithmetic information.

And if f is full-integral weight (and a Hecke eigenform), then $L(s,f)$ has an Euler product

$$L(s,f) = \prod_p L_p(s,f)$$

(and is called an L-function).

Half-integral weight L-series don't have Euler products.

Zeroes

Riemann Hypothesis

In addition, it is conjectured that L-functions associated to full-integral weight cusp forms satisfy a Riemann Hypothesis, i.e. that all nontrivial zeroes are on the line $\operatorname{Re} s = \frac{1}{2}$.

Question

What about for half-integral weight L-series?

Do half-integral weight L-series satisfy the Riemann Hypothesis?

Numerical Investigation

Jointly with Thomas Hulse and Li-Mei Lim, we have begun to numerically compute and collect zeroes of half-integral weight L-series.

And short answer is no, half-integral weight L-series fail the Riemann Hypothesis.

This is already known in folklore, and the first computed counterexample that I'm aware of is from 1994, due to Yoshida [Yos94].[1]

If $\widetilde{f}$ is the (unique) cusp form of weight $9/2$ on $\Gamma_0(4)$, then

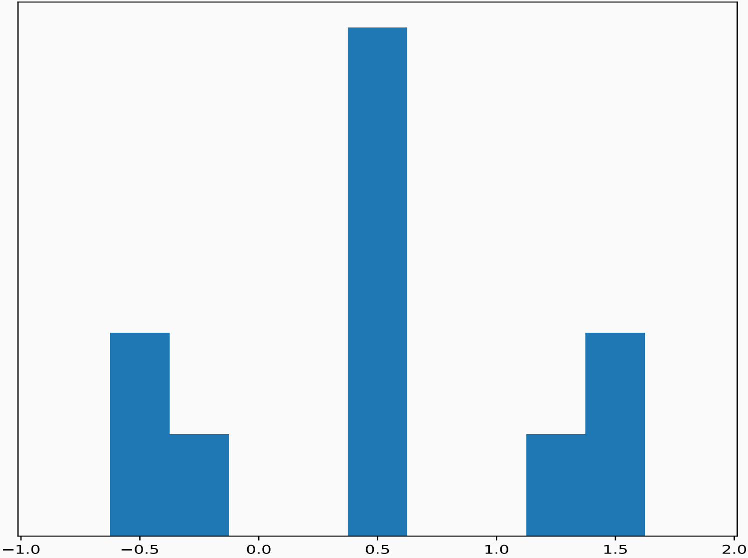
$$0.980820\ldots + 8.949629\ldots$$

is the lowest zero not on the critical line.

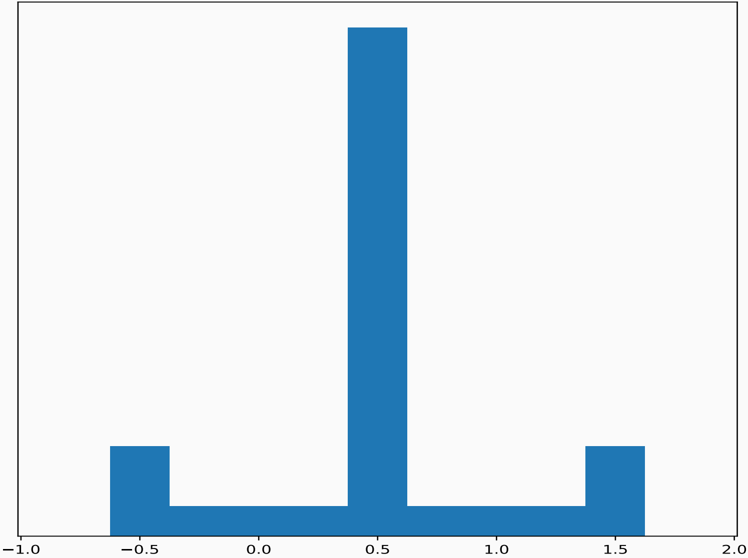
Let's look at the locations of more zeroes of this L-series.

[1]Which I must admit I wasn't aware of when beginning my calculations

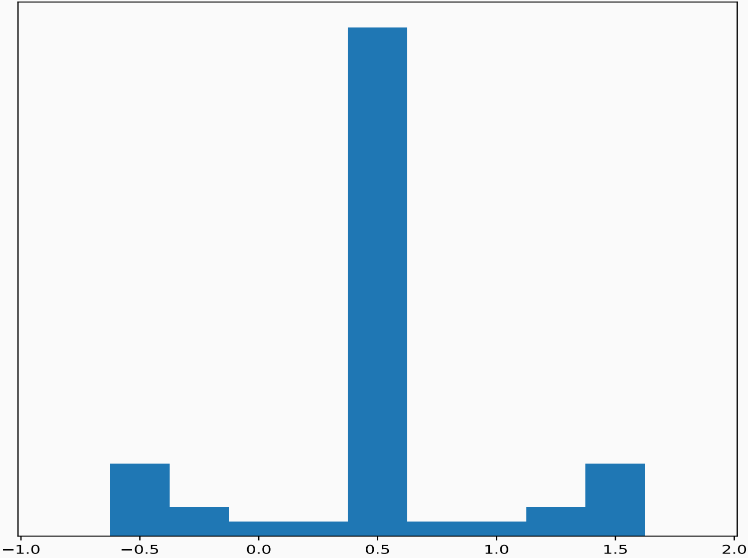
Zeroes with $\operatorname{Im} s < 25$



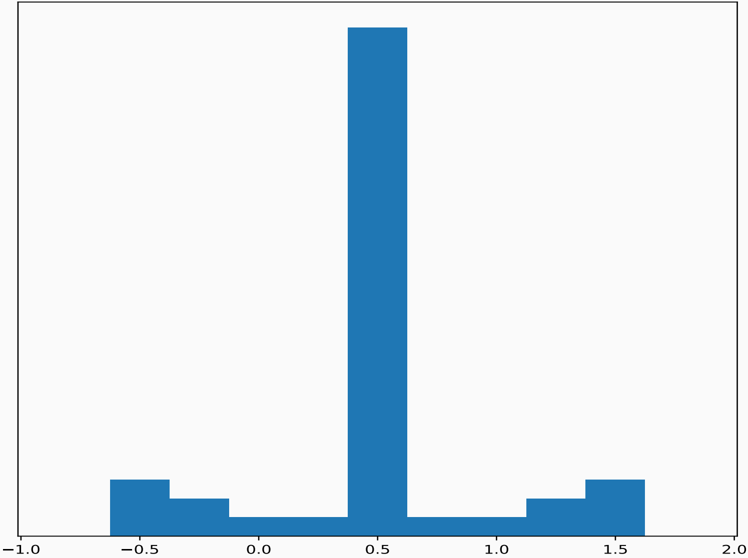
Zeroes with $\operatorname{Im} s < 50$



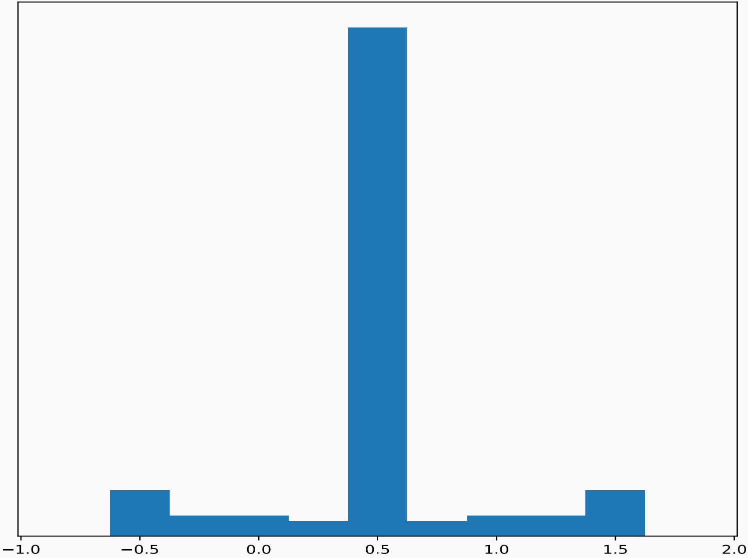
Zeroes with $\operatorname{Im} s < 75$



Zeroes with $\operatorname{Im} s < 100$



Zeroes with $\operatorname{Im} s < 150$



In fact, most of the zeroes that appear are exactly on the critical line. As one looks at more and more zeroes, it appears that a greater and greater percentage of the zeroes occur on the critical line.

Can we prove this?

Let $N(T)$ denote the number of zeroes with imaginary part between T and $2T$. Then $N(T) \sim cT \log T$ (just as for $\zeta(s)$ — the same proof applies).

More generally, let $N_a(T)$ denote the number of roots of $L(s,f) - a = 0$ with $T \le \operatorname{Im} s < 2T$. [And suppose that $a(1) = 1$, and that $a \ne 1$].

Levinson [Lev74] proved that all but $N_a^\zeta(T)(\log\log T)^{-1}$ of the solutions to $\zeta(s) - a = 0$ lie within $(\log\log T)^2/\log T$ of the critical line. That is, 100 percent of the times the zeta function takes almost any value, that spot is very near the critical line.

Steuding [Ste03] extended this to the Selberg class of L-functions (and assuming the Lindelöf Hypothesis, which says that $L(1/2+it,f) \ll [c(f)(1+|t|)]^\epsilon$).

If one assumes the Ramanujan-Petersson conjecture for half-integral weight modular forms (which says roughly that the coefficients are all about the average expected size), then one can use Steuding's argument to show that 100 percent of the solutions of $L(s,\widetilde{f}) = a$ are within ϵ of the critical line for any $\epsilon > 0$.

Thus the “spike” that appears in the histograms appears to demonstrate this fact, that most zeroes occur on or very near the critical line.

You may notice that Steuding’s argument relies on the Lindelöf Hypothesis. Once might expect that since the Riemann Hypothesis is false for half-integral weight L-functions, then perhaps the Lindelöf Hypothesis is false.

But numerical experimentation (and other work in progress with Jeff Hoffstein, Min Lee, and others) suggests that the Lindelöf Hypothesis seems to be true for half-integral weight L-functions.

Thank you very much.

Please note that these slides (and references for the cited works) are available on my website (davidlowryduda.com).

References i

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