Counting Lattice Points on Hyperboloids

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One-Sheeted Hyperboloids

A one-sheeted hyperboloid $\mathcal{H}_d(h)$ is described by the equation

$$X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h,$$

where $h > 0$. Of particular interest is the 3-dimensional hyperboloid $\mathcal{H}_3(h)$, which is given by

$$X^2 + Y^2 = Z^2 + h.$$

One might ask how many lattice points are on these hyperboloids? *(Answer: infinitely many).*

How many lattice points are on these hyperboloids, and are not too large? This is our guiding question.

Counting points on one-sheeted hyperboloids

Let $h \in \mathbb{Z}^+$. How many lattice points are on the surface of the hyperboloid $\mathcal{H}_d(h)$ *and* contained within the ball $B(\sqrt{R})$ of radius $\sqrt{R}$?

This question is similar in flavor to the Generalized Gauss Circle Problem of counting the number of lattice points within the d-dimensional ball $B(\sqrt{R})$, except that in our problem we impose an additional constraint.

As in the Generalized Gauss Circle Problem, the Hardy-Littlewood circle method can be applied for sufficiently large dimension. But smaller dimensions are more challenging. And the 3-dimensional case is by far the most enigmatic.

Oh and Shah [OS11] recently showed using ergodic techniques to show that the number of lattice points on

$$X^2 + Y^2 = Z^2 + h^2$$

and within $B(\sqrt{R})$ is

$$C\sqrt{R}\log R + O(R^{\frac{1}{2}}(\log R)^{\frac{3}{4}})$$

for an explicit constant C depending on h. Producing a better asymptotic and understanding the case when h is not a square has proved challenging.

A New Approach

Note that if $X^2 + Y^2 = Z^2 + h$, then the condition of being contained in $B(\sqrt{R})$ is the same as

$$(X^2 + Y^2) + Z^2 \leq R \iff 2Z^2 + h \leq R.$$

By considering each possible value of $Z^2 + h$ separately, we see that the number of points on $\mathcal{H}_3$ and within $B(\sqrt{R})$ is given by

$$\sum_{2n^2+h \leq R} r_2(n^2 + h) \approx \frac{1}{2} \sum_{2n+h \leq R} r_2(n + h) r_1(n),$$

where $r_k(n)$ is the number of ways of representing n as a sum of k squares.

These sums appear as Perron-type integrals of the Dirichlet series

$$\sum_{n \geq 0} \frac{r_2(n^2 + h)}{(2n^2 + h)^s} \approx \frac{1}{2} \sum_{n \geq 0} \frac{r_2(n + h) r_1(n)}{(2n + h)^s}.$$

If we can understand this Dirichlet series, we can produce asymptotics.

Modular Forms

The key insight is that this Dirichlet series can also be retrieved from a modular form. Let

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = 1 + \sum_{n \geq 1} r_1(n) e^{2\pi i n z}$$

be the standard Jacobi theta function, a modular form of weight $1/2$ on $\Gamma_0(4)$. (Note also that $\theta^d(z) = 1 + \sum_{n \geq 1} r_d(n) e^{2\pi i n z}$).

The relevant modular form for $\mathcal{H}_3$ is $V(z) = \theta^2(z)\overline{\theta(z)}$. (And for $\mathcal{H}_d$, it's $\theta^{d-1}(z)\overline{\theta(z)}$).

In particular, the hth Fourier coefficient of $V(z)$ is given by

$$\sum_{n \in \mathbb{Z}} r_2(n^2 + h) e^{-(2n^2+h)\pi y},$$

which is an exponentially weighted version of the sum we want to understand.

Methodology

Let $P_h^k(z,s)$ be a weight k Poincaré series,

$$P_h^k(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \mathrm{Im}(\gamma z)^s e^{2\pi i h \gamma z} J(\gamma, z)^k.$$

The Petersson inner product of $P_h^{\frac{1}{2}}(z,s)$ against V gives

$$\langle P_h^{\frac{1}{2}}(z,s), V(z) \rangle = \frac{\Gamma(s - \frac{1}{4})}{(2\pi)^{s-\frac{1}{4}}} \sum_{m \in \mathbb{Z}} \frac{r_2(m^2+h)}{(2m^2+h)^{s-\frac{1}{4}}},$$

which is our Dirichlet series (and an easily understood analytic factor). We have reduced the task to understanding the inner product on the left.

Roughly speaking, we understand the inner product by spectrally decomposing the Poincaré series into a sum of Maass forms and Eisenstein integrals, and understanding each of these terms separately. This is too technical for polite conversation, but additional details can be seen in Chapter 5 of my thesis [LD17].

The end result is that we get a complete meromorphic continuation to $\mathbb{C}$ of the series

$$\sum_{m\in\mathbb{Z}} \frac{r_{d-1}(m^2+h)}{(2m^2+h)^s}.$$

and it is possible to use this Dirichlet series to prove a variety of results.

Theorem

The number of integer lattice points on the hyperboid $\mathcal{H}_3$ and within the ball of radius $\sqrt{R}$ centered at the origin is

$$\sum_{2n^2+h\leq R} r_2(n^2+h) = \delta_{[h=a^2]} C' R^{\frac{1}{2}} \log R + CR^{\frac{1}{2}} + O(R^{\frac{1}{2}-\frac{1}{44}+\epsilon}).$$

More generally, for $\mathcal{H}_d(h)$ and $d \geq 4$, we have

$$\sum_{2n^2+h\leq R} r_d(n^2+h) = CR^{\frac{d-2}{2}} + O(R^{\frac{d-2}{2}-\lambda+\epsilon})$$

for a computable $\lambda = \lambda(d) > 0$.

Thank you very much.

Please note that these slides are available on my website, davidlowryduda.com.

References i

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