On Some Problems Related to the Gauss Circle Problem

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Introduction

The Gauss Circle Problem

In this talk, we will be discussing analogies and variants of the Gauss Circle Problem.

Gauss Circle Problem

How many integer lattice points are contained in a circle or radius $\sqrt{R}$ centered at the origin? Or equivalently: how many integer solutions are there to $x^2 + y^2 \leq R$?

We will use $S_2(R)$ to denote the number of integer lattice points inside the circle or radius $\sqrt{R}$.

Let $r_k(m)$ denote the number of k-tuples $(n_1, n_2, \ldots, n_k)$ such that $n_1^2 + \cdots + n_k^2 = m$. Then the Gauss Circle Problem is also equivalent to estimating

$$S_2(R) := \sum_{0 \leq m \leq R} r_2(m).$$

This is a classical problem, first considered over 200 years ago.

It is intuitively clear that $S_2(R) \approx \operatorname{Vol} B(\sqrt{R})$. This was known to Gauss, who showed that

$$S_2(R) - \operatorname{Vol} B(\sqrt{R}) \ll \sqrt{R},$$

or rather that the discrepancy between the number of lattice points and the area is at most the perimeter (up to maybe some constant).

Perhaps the first improvement came from Sierpiński [Sie06], who showed that

$$S_2(R) - \operatorname{Vol} B(\sqrt{R}) \ll R^{\frac{1}{3}}.$$

The best current bound is due to Heath-Brown [HB99], who showed

$$S_2(R) - \operatorname{Vol} B(\sqrt{R}) \ll R^{\frac{131}{416}+\epsilon}.$$

What is the correct bound?

Mean Square Estimates

Hardy and Littlewood showed that *on average*, the correct exponent is $\frac{1}{4}$.That is, they showed that

$$\int_0^X \left|S_2(r) - \operatorname{Vol} B(\sqrt{r})\right|^2 dr = cX^{\frac{3}{2}} + O(X^{\frac{5}{4}+\epsilon}),$$

and also that

$$S_2(R) - \operatorname{Vol} B(\sqrt{R}) = \Omega(R^{\frac{1}{4}}),$$

so that the "correct" order of growth appears to be $\frac{1}{4}$ in the exponent.

Enter Ramanujan

It is an interesting coincidence that Ramanujan was studying the coefficients of the Ramanujan τ function, defined by equating coefficients in

$$\sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

Ramanujan believed that

$$\tau(n) \ll n^{\frac{11}{2} + \epsilon}.$$

(This turned out to be true, though it's a bit hard to guess why Ramanujan thought so).

It was later also conjectured that

$$\sum_{n \leq X} \tau(n) \ll X^{\frac{11}{2} + \frac{1}{4} + \epsilon}.$$

The conjectured $\frac{1}{4}$ is very reminiscent of the Circle Problem.

Connections to Modular Forms

If one assembles the Ramanujan τ function as

$$\Delta(z) = \sum_{n \geq 1} \tau(n) e(nz)$$

then we get the Δ function, which is a weight 12 modular cusp form on $\mathrm{SL}(2, \mathbb{Z})$.It turns out that the observations on Ramanujan's τ function generalize towards many modular forms.

Suppose $f(z) = \sum a(n)e(nz)$ is a weight k cusp form on GL(2). Then it is now a celebrated theorem of Deligne [Del74] that

$$a(n) \ll n^{\frac{k-1}{2}+\epsilon},$$

and conjectured that

$$\sum_{n \leq X} a(n) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

Let $S_f(n)$ denote the partial sum of the first n Fourier coefficients of a weight k modular cusp form $f(z)$,

$$S_f(n) := \sum_{m \leq n} a(m).$$

Then Chandrasakharan and Narasimhan [CN62] showed a mean square estimate parallel to that of Hardy and Littlewood:

$$\int_0^X \left|S_f(t)\right|^2 dt = cX^{k-1+\frac{3}{2}} + O(X^{k+\epsilon}).$$

And it is now conjectured that the "correct order of growth" of $S_f(n)$ is also $\frac{1}{4}$.

Cusp Form Analogy

The goal of finding the correct order of growth for the size of $S_f(n)$ is what I call the "Cusp Form Analogy" to the Gauss Circle Problem.

Progress on the Cusp Form Analogy

Dirichlet Series for Sums of Coefficients of Cusp Forms

The state of the art right with respect to the Cusp Form Analogy for an individual sum is essentially due to Hafner and Ivić [HI89], who showed that (with some restrictions on f)

$$S_f(n) \ll n^{\frac{k-1}{2}+\frac{1}{3}+\epsilon}.$$

For mean-square results, there has been little progress since Chandrasakharan and Narasimhan.However, there has been progress on "short-interval estimates," culminating in Jutila's proof [Jut87] that

$$\frac{1}{X^{\frac{3}{4}+\epsilon}} \sum_{|n-X|\leq X^{\frac{3}{4}+\epsilon}} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}.$$

This says that *the conjectured bound holds on intervals of length* $X^{\frac{3}{4}+\epsilon}$ *around* X, and is qualitatively stronger than the mean-square result.

Why should we care about short-interval type results?

It is possible to easily prove estimates for individual $S_f(n)$ from short-interval averages of $S_f(n)$.In particular, if

$$\frac{1}{X^\alpha} \sum_{|n-X|<X^\alpha} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}},$$

then one can show that

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{6}+\frac{\alpha}{3}}.$$

So the Conjecture would follow from a short-intervals result over an interval of length $X^{\frac{1}{4}}$ around X.

(How do we show this? Briefly, suppose there is are large X for which $S_f(X) \geq X^{\frac{k-1}{2}+\beta}$. Then $S_f(X+\ell) \gg S_f(X)$ for $\ell \ll X^\beta$, since each summand is much smaller than the whole sum. Now estimate $\sum_{|n-X|\leq X^\beta} |S_f(X)|^2$ and compare to the short-interval result.)

Along with my collaborators Hulse, Kuan, and Walker, we investigated this short-interval question. [HKLDW17b]

Theorem (HKLDW II)

$$\frac{1}{X^{\frac{2}{3}}(\log X)^{\frac{1}{6}}} \sum_{|n-X| \leq X^{\frac{2}{3}}(\log X)^{\frac{1}{6}}} |S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}.$$

This is a sizable improvement over Jutila's $\frac{3}{4}$. As an easy corollary, we can show that

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{7}{18}},$$

(which is worse than what's known). With a lot of extra work, we can show that

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{3}},$$

(which matches what's known, up to a tiny log factor). So this isn't quite strong enough to improve individual bounds.

What's the New Idea?

In a set of recent papers (and my thesis), three new Dirichlet series were introduced and studied:

$$\sum_{n\geq 1} \frac{S_f(n)}{n^s}, \quad \sum_{n\geq 1} \frac{|S_f(n)|^2}{n^s}, \quad \text{and} \quad \sum_{n\geq 1} \frac{S_f(n)^2}{n^s}.$$

These are new objects, and (as far as I can tell) the latter two don't have any good reason to behave nicely. However, each has (mostly understandable) meromorphic continuation to the plane. These are very natural Dirichlet series to use in the study of the Cusp Form Analogy.

At the heart of the analysis are shifted convolution sums in two complex variables,

$$Z(s, w) := \sum_{n,h} \frac{a(n+h)a(n)}{(n+h)^s n^w},$$

as well as the spectral theory of automorphic forms.

The obstacle to further improvement is that we know so little about the distribution of eigenvalues of the hyperbolic Laplacian, which obfuscates a detailed analysis of the discrete spectrum.

Brief Methodology

One can (roughly) decompose the Dirichlet series into

$$\sum_{n\geq 1} \frac{S_f(n)^2}{n^s} = L(s, f\times f) + \int_{(\sigma)} L(s-z, f\times f)\zeta(z)B(z, s-z)dz$$
$$+ Z(s,0) + \int_{(\sigma)} Z(s-z,0)\zeta(z)B(z,s-z)dz,$$

where $Z(s,w)$ is the convolution (from the previous slide) and $B(a,b)$ is the Beta function.This reduces the study to a sufficient analytic understanding of $L(s, f\times f)$ and $Z(s,w)$.

This decomposition follows from a Mellin-Barnes type integral identity,

$$\sum_{n,m\geq 1} \frac{a(n)^2}{(n+m)^s} = \sum_{n,m\geq 1} \int_{(\sigma)} \frac{a(n)^2}{n^{s-z}} \frac{1}{m^z} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)} dz.$$

Brief Methodology

With $P_h(z,s)$ as a Poincaré series, we can understand the shifted convolution

$$Z(s,w) = \sum_{h \geq 1} \frac{\langle P_h(z,s), |f|^2 \rangle}{h^w}$$

by using Selberg's Spectral Decomposition on the Poincaré series.

That is, write

$$P_h(z,s) = \underbrace{\sum_j \langle P_h, \mu_j \rangle \mu_j(z)}_{\text{Discrete Spectrum}} + \underbrace{\sum \int_{(1/2)} \langle P_h, E(\cdot,u) \rangle E(z,u) du}_{\text{Continuous Spectrum}},$$

and substitute into the expression for $Z(s,w)$.The great challenge is a lack of understanding of the discrete spectrum.

Further Results

Using these new Dirichlet series, my collaborators and I were able to prove the following smoothed mean square result [HKLDW17a].

Theorem (HKLDW I)

$$\sum_{n\geq 1} |S_f(n)|^2 e^{-n/X} = CX^{k-1+\frac{3}{2}} + O(X^{k-1+\frac{1}{2}+\epsilon}).$$

Actually, we prove something a bit mysterious. If g is another weight k cusp form, we show

Theorem (HKLDW I)

$$\sum_{n\geq 1} S_f(n)\overline{S_g(n)} e^{-n/X} = C'X^{k-1+\frac{3}{2}} + O(X^{k-1+\frac{1}{2}+\epsilon}).$$

Theorem (HKLDW I restated)

$$\sum_{n\geq 1} S_f(n)\overline{S_g(n)}e^{-n/X} = C'X^{k-1+\frac{3}{2}} + O(X^{k-1+\frac{1}{2}+\epsilon}).$$

This says that the sums $S_f(n)$ and $S_g(n)$ correlate very strongly (which is surprising since both are changing signs with high frequency).In fact, the sign changes of $S_f(n)$ (and the individual $a(n)$) are closely related to the Cusp Form Analogy.

This is related to, but a bit different from, the Sato-Tate description of the distribution of the sizes of individual $a(n)$.

Since each $a(n) \sim n^{\frac{k-1}{2}+\epsilon}$, if the signs of the individual $a(n)$ were merely random, then we would expect *square root cancellation*,

$$S_f(X) \sim X^{\frac{k-1}{2}+\frac{1}{2}}.$$

Since we get *strictly more cancellation*, it's as though the individual $a(n)$ collude to make the sum very small!

Along this train of thought, one can prove [HKLDW17c]

Theorem (HKLDW III)

The sequence $\{S_f(n)\}_{n\in\mathbb{N}}$ has at least one sign change for some n in the interval $[X, X + X^{\frac{2}{3}+\epsilon}]$ for all $X \gg 1$.

Actually, we can again prove something very mysterious. We can also show that the *overnormalized sums* $\sum a(n)/n^{\frac{k-1}{2}+\frac{1}{4}}$ change sign regularly, indicating that the $a(n)$ really do "collude" to cancel as much as possible.

The Gauss Sphere Problem

A different variation of the Gauss Circle Problem is to instead ask:

Gauss d-Sphere Problem

How many integer lattice points are contained in a d-sphere of radius $\sqrt{R}$ centered at the origin? Or equivalently: how many integer solutions are there to $x_1^2 + \cdots + x_d^2 \leq R$?

We will use $S_d(2)$ to denote the number of lattice points inside the sphere of radius $\sqrt{R}$. Further, note that the Gauss d-Sphere Problem is equivalent to estimating

$$S_d(R) = \sum_{0 \leq m \leq R} r_d(m).$$

This is also a classical and highly studied problem.

As with the Circle Problem, it is intuitively clear that $S_d(R) \approx \operatorname{Vol} B_d(\sqrt{R})$, so the real goal is to understand $|S_d(R) - \operatorname{Vol} B_d(\sqrt{R})|$. Through a Gauss-like argument, one can show that

$$S_d(R) - \operatorname{Vol} B_d(\sqrt{R}) \ll R^{\frac{d-1}{2}},$$

bounding the error by the surface area. But just as in the Circle Problem, something stronger is conjectured [IKKN04]:

$$S_d(R) - \operatorname{Vol} B_d(\sqrt{R}) \ll R^{\alpha(d)}, \text{where } \alpha(d) = \begin{cases} \frac{1}{4} & d = 2 \\ \frac{d}{2} - 1 & d \geq 3. \end{cases}$$

Notice the phase shift between dimensions 2 and 3. This somehow reflects that these two dimensions are the most enigmatic.

What's Known for Higher Dimensions?

For really high dimensions, the circle method can be used to give very accurate estimates.In 3 dimensions, the state of the art for an individual estimate is due to Heath-Brown [HB99], and says

$$S_3(R) - \operatorname{Vol} B_3(\sqrt{R}) \ll R^{\frac{21}{32}+\epsilon}.$$

In 4 dimensions, the task is much easier since $r_4(n) = 8\sigma(n) - 32\sigma(\frac{n}{4})$, which is multiplicative and relatively well-behaved. One can show

$$S_4(R) - \operatorname{Vol} B_4(\sqrt{R}) \ll R \log R,$$

which is only a log power off.In dimensions $d \geq 5$, the correct order of magnitude is known for the discrepancy.

Mean Square Estimates

Mean square results follow a similar pattern, and are understood very well for $d \geq 6$. Of particular interest is the enigmatic 3-dimensional case. Jarnik [Jar40] gave essentially the only major progress (in 1940) when he showed that

$$\int_0^X |S_3(r) - \operatorname{Vol} B_3(\sqrt{r})|^2 dr = CX^2 \log X + O(X^2 (\log X)^{\frac{1}{2}}).$$

Note that the main term comes with a log factor — the 3 dimensional case is unique in this regard.

When $d = 4$, there is a power savings of $X^{\frac{1}{2}}$ (ignoring log factors), and for $d \geq 5$ there is a power savings of X (also ignoring log factors). So among all the d-dimensional Gauss Sphere Problems, the case when $d = 3$ is the least understood.

Idea: Build More Dirichlet Series

In analogy with the methodology for cusp forms, my collaborators and I set out to investigate the Dirichlet series

$$\sum_{n\geq 0} \frac{S_3(n)^2}{n^s} \quad \text{and} \quad \sum_{n\geq 0} \frac{(S_3(n) - \operatorname{Vol} B_3(\sqrt{n}))^2}{n^s}.$$

The thought is that $S_3(n)$ are the sums of the coefficients of the modular form $\theta^3(z)$, so maybe a similar construction will work. Miraculously, this *does work*, and the ideas are very similar.

The primary difficulty is to get a deep understanding of the shifted convolution sum

$$\sum_{n,h\geq 1} \frac{r_3(n+h)r_3(n)}{(n+h)^s n^w},$$

and most of the ideas carry forward.

With two key exceptions.

θ^3 is a Half-Integral Weight Non-Cuspform

The underlying modular form is

$$\theta^3(z) = \Big(\sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} \Big)^3 = 1 + \sum_{n \geq 1} r_3(n) e^{2\pi i n z}.$$

This is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4)$, and is not cuspidal.

- Half-integral weight modular forms carry a large set of challenges, largely due to the fact that their coefficients are not multiplicative.
- Further, a complete spectral analysis is much simpler with cuspforms, so it is necessary to modify our modular forms by subtracting other, well-understood (i.e. Eisenstein) modular forms to get a "small" object for analysis.

Fruits of Labor

In a preprint that will appear on the arXiv this week (or maybe next week), we show the following theorem.

Theorem

There exists $\lambda > 0$ *such that*

$$\int_0^X \left|S_3(t) - \operatorname{Vol} B_3(\sqrt{t})\right|^2 dt = C' X^2 \log X + CX^2 + O(X^{2-\lambda+\epsilon}).$$

This breaks Jarnik's X^2 barrier and extracts a second main term.

Pushing our analysis to its extremes, we believe we can actually prove

Claim

$$\int_0^X \left|S_3(t) - \operatorname{Vol} B_3(\sqrt{t})\right|^2 dt = C' X^2 \log X + CX^2 + O(X^{2-\frac{1}{5}+\epsilon}).$$

More generally, we also consider smoothed mean square estimates. With smoothing, it is possible to see many minor terms.

Theorem

$$\sum_{n\geq 1}\left|S_d(n) - \operatorname{Vol} B_d(\sqrt{n})\right|^2 e^{-n/X}$$
$$= \delta_{[d=3]} C' X^2 \log X + C_d X^{d-1}$$
$$+ \delta_{[d=4]} C_4 X^{\frac{5}{2}} + C_d'' X^{d-2} + O(X^{d-\frac{5}{2}+\epsilon}).$$

Notice that the $d = 4$ case is unique in that there is a second term one half power of X below the main term.

As far as I know, this is the first theorem of this type for the d-dimensional Sphere Problem. Combined with numerical experimentation, it appears that there are very often secondary main terms. It would be interesting to know what to expect in this situation, but we are still uncertain.

One-Sheeted Hyperboloids

The d-dimensional Gauss Sphere Problem concerns counting

$$\#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 \leq R\} = \sum_{m \leq R} r_d(m).$$

Suppose instead we want to count the number of lattice points on the one-sheeted hyperboloid $\mathcal{H}_{d,h}$ for some positive integer h,

$$\#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_{d-1}^2 = x_d^2 + h\}.$$

(Answer: infinitely many). So let's count the number of lattice points on the one-sheeted hyperboloid $\mathcal{H}_{d,h}$ *and* inside the ball $B(\sqrt{R})$. This is equivalent to counting

$$\sum_{2m^2 + h \leq R} r_{d-1}(m^2 + h),$$

which looks very similar to the Gauss $d-1$ Sphere Problem sum, except constrained along a quadratic.

This leads to the following question (which is certainly in the same flavor of problem as the Gauss Circle Problem).

One-Sheeted Hyperboloid Problem

How many integer lattice points are contained within the d-dimensional sphere or radius $\sqrt{R}$ centered at the origin **and** on the one-sheeted hyperboloid

$$\mathcal{H}_{d,h} = X_1^2 + \cdots + X_{d-1}^2 = X_d^2 + h?$$

Equivalently, estimate the size of

$$\sum_{2m^2+h\leq R} r_{d-1}(m^2+h).$$

What's Known?

In many dimensions, the circle method should be able to determine a main term with some logarithmic savings, with better savings occurring for very high dimension.

The two dimensional case is now uninteresting, but the three-dimensional case is again very enigmatic. When h is a square, it is easy to come up with a heuristic. Consider

$$X^2 + Y^2 = Z^2 + h^2.$$

Then setting $X = Z, Y = h$ gives $\sqrt{R}$ trivial terms. It's natural to ask: Are these most of the solutions, or are we missing many more?

Oh and Shaw [OS11] recently showed that when h is a square, the total number of solutions is

$$C\sqrt{R}\log R + O(R^{\frac{1}{2}}(\log R)^{\frac{3}{4}}).$$

So we see that most solutions are nontrivial.

An Aside on Prime Paucity

We've seen that almost all solutions to

$$X^2 + Y^2 = Z^2 + h^2$$

do not come from setting X or Y to h (the "trivial" solutions). It is an interesting question to restrict the potential solutions to a sparser set, and to see what changes. For example, if X, Y, Z are restricted to primes and h is a fixed prime, then there are about $\sqrt{R}/(\log R)^2$ trivial solutions.

In a paper (preprint to appear later this spring) with Ayla Gafni and Sam Chow, we show that there are at most $\sqrt{R}/(\log R)^3$ non-trivial solutions — so in this case, almost all solutions *are* trivial! This extends a result of Erdös, and we are considering other cases when restricting to thin sets changes the characteristics of the set of solutions.

Central Idea

We return now to the standard hyperboloid $X^2 + Y^2 = Z^2 + h$, when h is not necessarily a square. The key idea is that these solutions can also be retrieved from a modular form, namely $V(z) = \theta^2(z)\overline{\theta(z)}$.

In particular, the hth Fourier coefficient of $V(z)$ is given by

$$\sum_{m \in \mathbb{Z}} r_2(m^2 + h) e^{-(2m^2+h)\pi y},$$

which is an exponentially weighted version of the sum we want to understand. By taking an inner product against a weight $\frac{1}{2}$ Poincaré series, we can recover the Dirichlet series

$$\langle P_h^{\frac{1}{2}}(z, s), V(z) \rangle \sim \sum_{m \in \mathbb{Z}} \frac{r_2(m^2 + h)}{(2m^2 + h)^s}.$$

This Dirichlet series is analogous to the shifted convolution sums $Z(s,w)$ studied in the previous variants of the Gauss Circle Problem. And we can try to understand it using the same methodology: spectrally expand the Poincaré series, produce a meromorphic continuation, and then use classical complex analytic methods (like Perron's formula).

But now there are a host of difficulties.

- $\theta^2(z)\overline{\theta(z)}$ is neither holomorphic nor cuspidal, both of which cause technical difficulties.
- The Poincaré series $P_h^{\frac{1}{2}}(z,s)$ is half-integral weight, which means that its spectral expansion is significantly more complicated and mysterious.
- To use the spectral expansion, we want to subtract Eisenstein series from $V(z)$ to make a function $\widetilde{V}(z)$ which is square integrable, and it just so happens that the necessary Eisenstein series are evaluated at a pole, so one must further adjust the methodology.

Unfortunately, all of these problems are very technical and do not lend themselves to a talk. But It's worth mentioning that it is possible to attain the meromorphic continuation for the Dirichlet series

$$\sum_{m \in \mathbb{Z}} \frac{r_{d-1}(m^2+h)}{(2m^2+h)^s}.$$

and it is possible to use this Dirichlet series to prove a variety of results.

Of particular interest is the following (which appears in my thesis).

Theorem

The number of integer lattice points on the hyperboid $\mathcal{H}_{d,h}$ and within the ball of radius $\sqrt{R}$ centered at the origin is

$$\delta_{[h=a^2]} C' R^{\frac{1}{2}} \log R + C R^{\frac{1}{2}} + O(R^{\frac{1}{2}-\frac{1}{20}+\epsilon}).$$

As a corollary, note that when h is not a square, a positive proportion of solutions are trivial solutions!

Takeaways

An underlying theme to this talk is that there are still very many classical analogues of the Gauss Circle Problem which can be further understood by choosing the correct modular form and studying its coefficients.

Further, the theory of multiple Dirichlet series and shifted convolution sums play a tremendously important role. Because of this, a better understanding of the spectrum of the hyperbolic Laplacian can have a large effect on the understanding of these classical results.

Finally, isn't it cool that the Dirichlet series

$$\sum_{n \geq 1} \frac{S_f(n)^2}{n^s}$$

has a meromorphic continuation? I think it's pretty cool.

Thank you very much.

Are there any questions?

References I

K. Chandrasekharan and Raghavan Narasimhan.
Functional equations with multiple gamma factors and the average order of arithmetical functions.
Ann. of Math. (2), 76:93–136, 1962.

Pierre Deligne.
La conjecture de Weil. I.
Inst. Hautes Études Sci. Publ. Math., (43):273–307, 1974.

D. R. Heath-Brown.
Lattice points in the sphere.
In *Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997)*, pages 883–892. de Gruyter, Berlin, 1999.

References II

James Lee Hafner and Aleksandar Ivić.
On sums of Fourier coefficients of cusp forms.
Enseign. Math. (2), 35(3-4):375–382, 1989.

Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
The second moment of sums of coefficients of cusp forms.
J. Number Theory, 173:304–331, 2017.

Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
Short-interval averages of sums of Fourier coefficients of cusp forms.
J. Number Theory, 173:394–415, 2017.

Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker.
Sign changes of coefficients and sums of coefficients of L-functions.
J. Number Theory, 177:112–135, 2017.

A. Ivic, E. Krätzel, M. Kühleitner, and W. Nowák.
Lattice points in large regions and related arithmetic functions: Recent developments in a very classic topic.
2004.
http://arxiv.org/pdf/math/0410522.pdf.

Vojtěch Jarník.
Über die Mittelwertsätze der Gitterpunktlehre. V.
Časopis Pěst. Mat. Fys., 69:148–174, 1940.

References IV

M. Jutila.
Lectures on a method in the theory of exponential sums, volume 80 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics.
Tata Institute of Fundamental Research, 1987.

Hee Oh and Nimish Shah.
Limits of translates of divergent geodesics and integral points on one-sheeted hyperboloids.
arXiv preprint arXiv:1104.4988, 2011.

W. Sierpiński.
O pewnem zagadnieniu z rachunku funckcyi asymptotycnych.
Prace mat.-fiz, pages 77–118, 1906.