

COMPUTING π WITH FIRST YEAR CALCULUS

DAVID LOWRY-DUDA

1. COMPUTING π

This note was originally written in the context of my fall Math 100 class at Brown University.

While investigating Taylor series, we proved that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots \quad (1)$$

Let's remind ourselves how. Begin with the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

(We showed that this has interval of convergence $|x| < 1$). Integrating this geometric series yields

$$\int_0^x \frac{1}{1+t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Note that this has interval of convergence $-1 < x \leq 1$.

We also recognize this integral as

$$\int_0^x \frac{1}{1+t^2} dt = \arctan(x),$$

one of the common integrals arising from trigonometric substitution. Putting these together, we find that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

As $x = 1$ is within the interval of convergence, we can substitute $x = 1$ into the series to find the representation

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.$$

Since $\arctan(1) = \frac{\pi}{4}$, this gives the representation for $\pi/4$ given in (1).

However, since $x = 1$ was at the very edge of the interval of convergence, this series converges very, very slowly. For instance, using the first 50 terms gives the approximation

$$\pi \approx 3.121594652591011.$$

The expansion of π is actually

$$\pi = 3.141592653589793238462\dots$$

So the first 50 terms of (1) gives two digits of accuracy. That's not very good.

I think it is very natural to ask: can we do better? This series converges slowly — can we find one that converges more quickly?

Aside. As an aside: one might also ask if we can somehow *speed up* the convergence of the series we already have. It turns out that in many cases, you can! For example, we know in alternating series that the sum of the whole series is between any two consecutive partial sums. So what if you took the average of two consecutive partial sums? [Equivalently, what if you added only one half of the last term in a partial sum. Do you see why these are the same?]

The average of the partial sum of the first 49 terms and the partial sum of the first 50 terms is actually

$$3.141796672793031,$$

which is correct to within 0.001. That's an improvement!

What if you do still more? More on this can be found in Section 4.

2. ESTIMATING π THROUGH A DIFFERENT SERIES

We return to the question: can we find a series that gives us π , but which converges faster? Yes we can! And we don't have to look too far — we can continue to rely on our expansion for $\arctan(x)$.

We had been using that $\arctan(1) = \frac{\pi}{4}$. But we also know that $\arctan(1/\sqrt{3}) = \frac{\pi}{6}$. Since $1/\sqrt{3}$ is closer to the center of the power series than 1, we should expect that the convergence is much better.

Recall that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Then we have that

$$\begin{aligned} \arctan\left(\frac{1}{\sqrt{3}}\right) &= \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} + \dots \\ &= \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \dots\right) \\ &= \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)3^n}. \end{aligned}$$

Therefore, we have the equality

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)3^n}$$

or rather that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)3^n}.$$

From a computation perspective, this is far superior. For instance, based on our understanding of error from the alternating series test, using the first 10 terms of this series will approximate π to within

$$2\sqrt{3} \frac{1}{23 \cdot 3^{11}} \approx \frac{1}{26680}.$$

Let's check this.

$$2\sqrt{3} \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} + \cdots + \frac{1}{21 \cdot 3^{10}} \right) = 3.1415933045030813.$$

Look at how close that approximation is, and we only used the first 10 terms! Roughly speaking, each additional 2.5 terms yields another digit of π . Using the first 100 terms would give the first 48 digits of π . Using the first million terms would give the first 47000 (or so) digits of π — and this is definitely doable, even on a personal laptop. (On my laptop, it takes approximately 4 milliseconds to compute the first 20 digits of π using this technique).

Even Better Series. I think it is very natural to ask again: can we find an even faster converging series? Perhaps we can choose better values to evaluate arctan at? This turns out to be a very useful line of thought, and it leads to some of the best-known methods for evaluating π . Through clever choices of values and identities involving arctangents, one can construct extremely quickly converging series for π . For more information on this line of thought, look up Machin-like formula.

3. PATTERNS IN THE APPROXIMATION OF $\pi/4$

Looking back at the approximation of π coming from the first 50 terms of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{2}$$

we found an approximation of π , which I'll represent as $\hat{\pi}$,

$$\pi \approx \hat{\pi} = 3.121594652591011.$$

Let's look very carefully at how this compares to π , up to the first 10 decimals. We color the incorrect digits in orange.

$$\begin{aligned} \pi &= 3.1415926535\dots \\ \hat{\pi} &= 3.1215946525 \end{aligned}$$

Notice that most of the digits are correct — in fact, only three (of the first ten) are incorrect! Isn't that weird?

It happens to be that when one uses the first $10^N/2$ terms (for any N) of the series (2), there will be a pattern of mostly correct digits with disjoint

strings of incorrect digits in the middle. This is an unusual and surprising phenomenon.

The positions of the incorrect digits can be predicted. Although I won't go into any detail here, the positions of the errors are closely related to something called *Euler Numbers* or, more deeply, to *Boole Summation*.

Playing with infinite series leads to all sorts of interesting patterns. There is a great history of mathematicians and physicists messing around with series and stumbling across really deep ideas.

4. SPEEDING UP COMPUTATION

Take an alternating series

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \dots$$

If $\{a_n\}$ is a sequence of positive, decreasing terms with limit 0, then the alternating series converges to some value S . And further, consecutive partial sums bound the value of S , in that

$$\sum_{n=0}^{2K-1} (-1)^n a_n \leq S \leq \sum_{n=1}^{2K} (-1)^n a_n.$$

For example,

$$1 - \frac{1}{3} < \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} < 1 - \frac{1}{3} + \frac{1}{5}.$$

Instead of approximating the value of the whole sum S by the K th partial sum $\sum_{n \leq K} (-1)^n a_n$, it might seem reasonable to approximate S by the average of the $(K-1)$ st partial sum and the K th partial sum. Since we know S is between the two, taking their average might be closer to the real result.

As mentioned above, the average of the partial sum consisting of the first 49 terms of (1) and the first 50 terms of (1) gives a much improved estimate of π than using either the first 49 or first 50 terms on their own. (And indeed, it works much better than even the first 500 terms on their own).

Before we go on, let's introduce a little notation. Let S_K denote the partial sum of the terms up to K , i.e.

$$S_K = \sum_{n=0}^K (-1)^n a_n.$$

Then the idea is that instead of using S_K to approximate the whole sum S , we'll use the average

$$\frac{S_{K-1} + S_K}{2} \approx S.$$

Averaging once seems like a great idea. What if we average again? That is, what if instead of using the average of S_{K-1} and S_K , we actually use

the average of (the average of S_{K-2} and S_{K-1}) and (the average of S_{K-1} and S_K),

$$\frac{\frac{S_{K-2}+S_{K-1}}{2} + \frac{S_{K-1}+S_K}{2}}{2}. \quad (3)$$

As this is really annoying to write, let's come up with some new notation. Write the average between a quantity X and Y as

$$[X, Y] = \frac{X + Y}{2}.$$

Further, define the average of $[X, Y]$ and $[Y, Z]$ to be $[X, Y, Z]$,

$$[X, Y, Z] = \frac{[X, Y] + [Y, Z]}{2} = \frac{\frac{X+Y}{2} + \frac{Y+Z}{2}}{2}.$$

So the long expression in (3) can be written as $[S_{K-2}, S_{K-1}, S_K]$.

With this notation in mind, let's compute some numerics. Below, we give the actual value of π , the values of S_{48} , S_{49} , and S_{50} , pairwise averages, and the average-of-the-average, in the case of $1 - \frac{1}{3} + \frac{1}{5} + \dots$.

	Value	Difference from π
π	3.141592653589793238462...	0
$4 \cdot S_{48}$	3.1207615795929895	0.020831073996803617
$4 \cdot S_{49}$	3.161998692995051	-0.020406039405258092
$4 \cdot S_{50}$	3.121594652591011	0.01999800099878213
$4 \cdot [S_{48}, S_{49}]$	3.1413801362940204	0.0002125172957727628
$4 \cdot [S_{49}, S_{50}]$	3.1417966727930313	-0.00020401920323820377
$4 \cdot [S_{48}, S_{49}, S_{50}]$	3.141588404543526	0.00000424904626727951

So using the average of averages from the three sums S_{48} , S_{49} , and S_{50} gives π to within $4.2 \cdot 10^{-6}$, an incredible improvement compared to S_{50} on its own.

There is something really odd going on here. We are not computing additional summands in the overall sum (1). We are merely combining some of our partial results together in a really simple way, repeatedly. Somehow, the sequence of partial sums contains more information about the limit S than individual terms, and we are able to extract some of this information.

I think there is a very natural question. What if we didn't stop now? What if we took averages-of-averages-of-averages, and averages-of-averages-of-averages-of-averages, and so on? Indeed, we might define the average

$$[X, Y, Z, W] = \frac{[X, Y, Z] + [Y, Z, W]}{2},$$

and so on for larger numbers of terms. In this case, it happens to be that

$$[S_{15}, S_{16}, \dots, S_{50}] = 3.141592653589794, \quad (4)$$

which has the first 15 digits of π correct!

By repeatedly averaging alternating sums of just the first 50 reciprocals of odd integers, we can find π up to 15 digits. I think that's incredible

— it seems both harder than it might have been (as this involves lots of averaging) and much easier than it might have been (as the only arithmetic input are the fractions $1/(2n + 1)$ for n up to 50).

Although we leave the thread of ideas here, there are plenty of questions that I think are now asking themselves. I encourage you to ask them, and we may return to this (or related) topics in the future. I'll see you in class.