Iterated Sums of Coefficients of Cusp Forms

Recent Results and New Conjectures

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Outline

Introduction

Sizes of coefficients of cusp forms

Let $f(z) = \sum_{n \geq 1} a(n) e(nz)$ be a full-integer weight k cusp form on a congruence subgroup of SL(2). The coefficients $a(n)$ are important and well-studied. One of the most fundamental questions we can ask is about their size.

Theorem (Deligne)

$$a(n) \ll n^{\frac{k-1}{2} + \epsilon}$$

It is also very natural to ask about their average order. Define

$$S_f(X) := \sum_{n \leq X} a(n).$$

What do we know about $S_f(X)$?

Sizes of sums of coefficients of cusp forms

Applying the Deligne bound naively to $S_f(X)$ leads to the bound

$$S_f(X) \ll X^{\frac{k-1}{2}+1+\epsilon}.$$

If we assume that the signs of the coefficients $a(n)$ are roughly random (or stronger, that they satisfy Sato-Tate), then we might expect the bound

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{2}+\epsilon}.$$

We actually expect an even better bound, analogous to the bounds in the Gauss Circle and Dirichlet Hyperbola problems.

Classical Conjecture (still a conjecture)

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

Recent Results Towards the Classical Conjecture

Theorem (Classical Conjecture on Average [CN64])

$$\frac{1}{X}\sum_{n\leq X}|S_f(n)|^2 = c_f X^{k-1+\frac{1}{2}} + O(X^{k-1+\epsilon})$$

Theorem (Smoothed Generalization [HKLDW15a])

$$\frac{1}{X}\sum_{n\geq 1} S_f(n)\overline{S_g(n)}e^{-n/X} = c_{f,g} X^{k-1+\frac{1}{2}} + O(X^{k-1-\frac{1}{2}+\epsilon}).$$

Theorem (Classical Conjecture in Short Intervals [HKLDW15b])

$$\frac{1}{X^{2/3}}\sum_{|n-X|<X^{2/3}}|S_f(n)|^2 \ll X^{k-1+\frac{1}{2}}.$$

Iterated Sums

Iterated Sums I

While investigating the Classical Conjecture, we showed that the Dirichlet series

$$\sum_{n\geq 1} \frac{S_f(n)}{n^s} \quad \text{and} \quad \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^s}$$

each are distinguished by having meromorphic continuation to the plane.

We wondered, what would happen if we looked at iterated partial sums? For $j \geq 0$, let

$$S_f^{(j+1)}(X) = \sum_{n\leq X} S_f^{(j)}, \qquad S_f^1(X) := S_f(X) = \sum_{n\leq X} a(n)$$

denote the iterated partial sums associated to f.

It is natural to ask again, how large are the $S_f^{(j)}$?

Iterated Sums II

As far as we can tell, this is a new question. Initial investigations into the properties of

$$\sum_{n\geq 1} \frac{S_f^{(j)}}{n^s}$$

and numerical experimentation suggest further remarkable properties and cancellation.

As a first attempt, one can $S_f^{(j)}(X)$ as a weighted sum of the individual coefficients $a(n)$ in the following way,

$$S_f^{(j)}(X) = \sum_{n\leq X} \binom{X-n+j-1}{j-1} a(n).$$

With this expression, it is easy to show that we can interpret $S_f^{(j)}(X)$ as a particular integral transform on $L(s,f)$. We can also show that $\sum S_f^{(j)} n^{-s}$ has meromorphic continuation.

An Example: $j = 2$

As an example, consider $S_f^{(2)}(X)$. Then

$$S_f^{(2)}(X) = \sum_{n \leq X} \binom{X - n + 1}{1} a(n) = \sum_{n \leq X} (X - n) a(n) + \sum_{n \leq X} a(n).$$

We recognize this as a sum of a standard cutoff integral transform and an inverse-Césaro weighted cutoff integral transform,

$$S_f^{(2)}(X) = \frac{1}{2\pi i} \int_{(\sigma)} L(s, f) \left(\frac{X^{s+1}}{s(s+1)} + \frac{X^s}{s} \right) ds.$$

In complete generality,

$$\sum_{n \geq 1} \frac{S_f^{(j)}(n)}{n^s} = L(s, f) + \frac{1}{2\pi i} \int_{(\sigma)} L(s - z, f) \zeta_j(z) B(z, s - z) dz$$

where $B(u, v)$ is the Beta function and $\zeta_j(z) = \sum_n \binom{n+j-1}{j-1} n^{-s}$ is a sort of binomial-coefficient zeta function.

Limits to this Approach

It is possible to say many partial results using these techniques. For instance, we can show that there is always at least squareroot-type cancellation [which we'll return to later]. But to say more, we would want to understand the squares $(S_f^{(j)}(X))^2$. Unfortunately, the techniques we have used so far do not extend to squares, and it's not obvious what the right answers *should* be.

For the rest of this talk, we'll look at some of the results of our experimentation to try to understand what the right conjectures should be.

Investigative Experimentation

A Note on Methodology and Reproducibility

My code and exact methodology are available on my website, `https://davidlowryduda.com`. I make extensive use of the free and open source SageMath[1], an excellent resource for mathematical (and number theoretic in particular) numerical exploration.

We now fix a choice of modular cusp form,

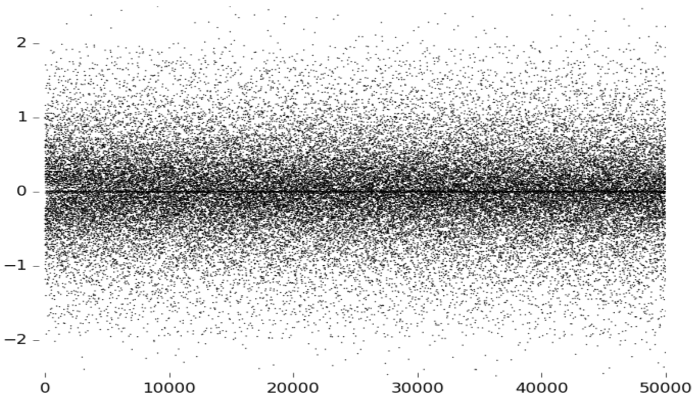
$$f(z) = \Delta(z) = \sum_{\tau(n)q^n} = q \prod_{n \geq 1} (1 - q^n)^{24},$$

the classical Discriminant function, i.e./ the weight 12 cusp form on $\mathrm{SL}(2, \mathbb{Z})$ whose coefficients are given by the Ramanujan τ function.

(We have performed similar numerical analysis on a variety of cusp forms and they present very similar trends).

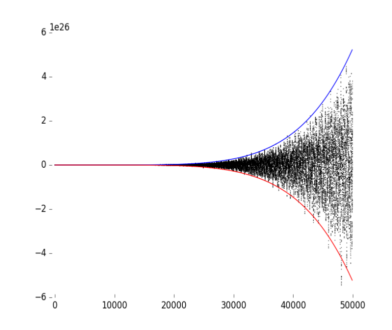
[1] See `https://www.sagemath.org` for more

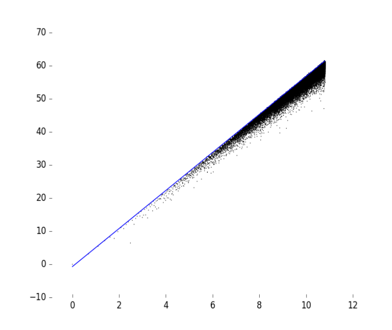
We collect the first few million coefficients of $f(z)$. For interest, this is what the first 50000 coefficients look like (after normalization — these are $\tau(n)/n^{11/2}$).



We can see apparently random signs (and apparent conformity to Sato-Tate).

We compute the partial sums $S_f(n)$ from these first few million coefficients. At the left, we show the first 50000 partial sums $S_f(n)$, along with the conjectured polynomial growth lines in red and blue. On the right, we show a log-log plot of the (absolute values of the) partial sums $S_f(n)$.





(Note how the Classical Conjecture seems very accurate).

Higher Iterated Moments I

What bounds should we expect for higher iterates? Clearly $S_f^{j+1}(X) \ll \sum_{n \leq X} |S_f^{(j)}(n)|$, leading to the estimates $S_f^{(j)} \ll X^{\frac{k-1}{2}+\frac{1}{4}+(j-1)+\epsilon}$. But should there be further cancellation, or perhaps remarkable cancellation as in the first moment?

It is known that $\{S_f(n)\}_{n\in\mathbb{N}}$ changes sign regularly.

Theorem ([HKLDW16])

For $X \gg 1$, there are at least $X^{1/3}$ sign-changes in $\{S_f(n)\}_{n\in\mathbb{N}}$ for $n \in [X, 2X]$.

So one might hope for repeated square-root cancellation, $S_f^{(j)} \ll X^{\frac{k-1}{2}+\frac{1}{4}+\frac{(j-1)}{2}+\epsilon}$. (In fact, we can already prove repeated square-root cancellation). Anything further would indicate regularity and structure in the sign-changes of the individual coefficients $a(n)$ which is beyond our understanding.

Higher Iterated Moments II

Numerically, we compute $S_f^{(j)}(n)$ for several n and find best-fit growth lines for the maximum sizes of $S_f^{(j)}$. These results (for the first 2.5 million n) are displayed for $j \leq 10$ at right.

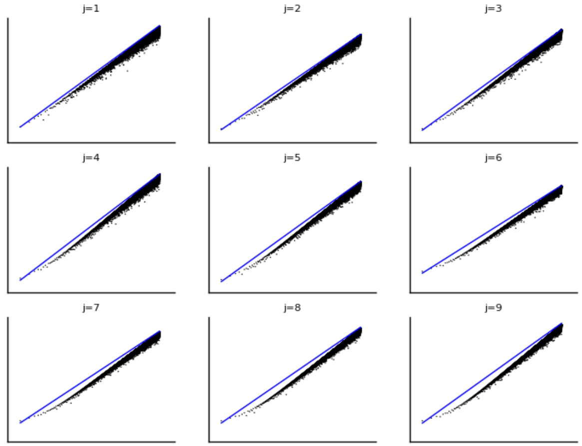
Notice for $j \leq 4$, the data is consistent with an iterated Classical Conjecture — each iteration contributes only $X^{1/4}$ or so to the sum over length X. There continues to be truly remarkable (and poorly understood) cancellation.
Caveat: we have ignored the presence of log factors. This means that the computed b are a bit too large.

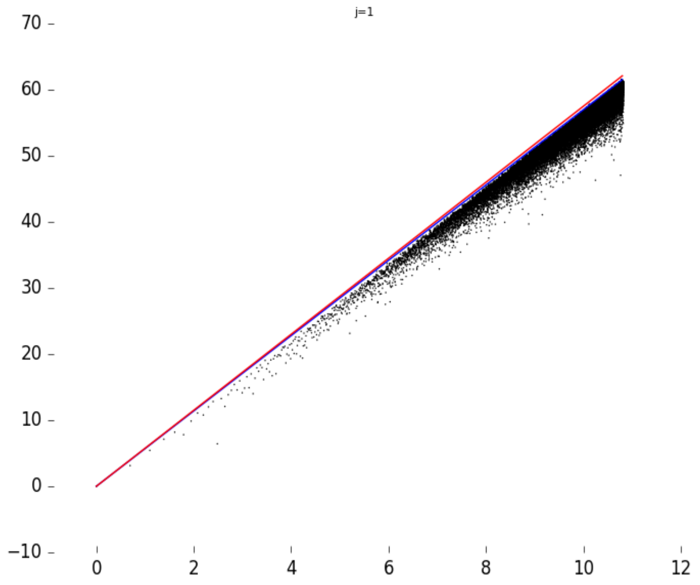
Best-fit approximations for $S_f^{(j)}(X) \ll X^b$

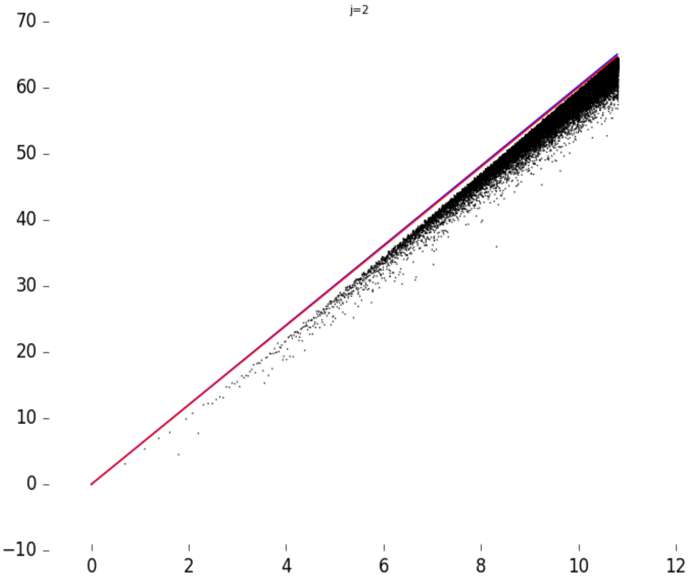
j	b	$b - \frac{11}{2}$	CC
0	5.58936	0.08936	0.0
1	5.67706	0.17706	0.25
2	5.94356	0.44356	0.5
3	6.24293	0.74293	0.75
4	6.55078	1.05078	1.0
5	6.86176	1.36176	1.25
6	7.17432	1.67432	1.5
7	7.48790	1.9879	1.75
8	7.80214	2.30214	2.0
9	8.11676	2.61676	2.25
10	8.43152	2.93152	2.5

Higher Iterated Moments III

Here are log-log plots of these iterated moment computations, with best-fit in blue.




j=1
70
60
50
40
30
20
10
0
−10
0
2
4
6
8
10
12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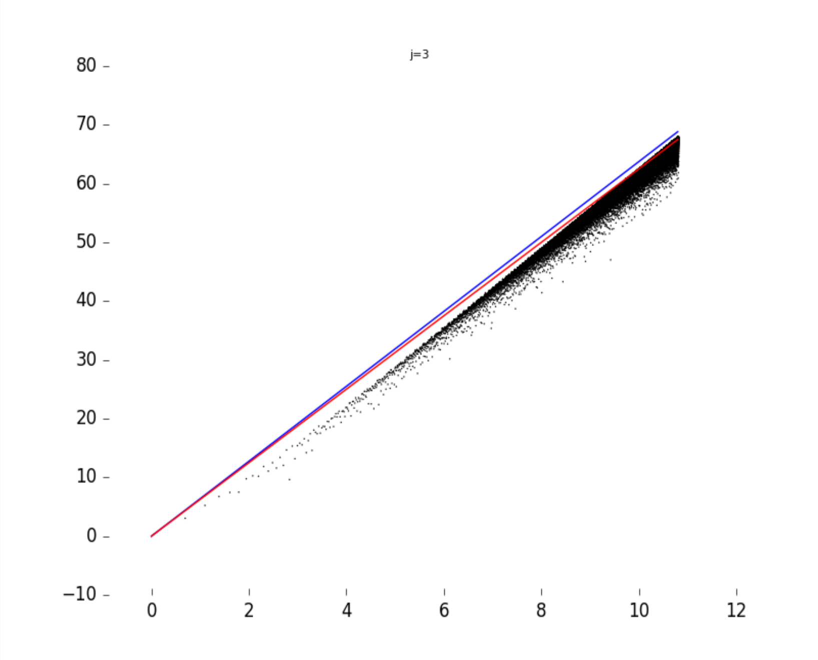
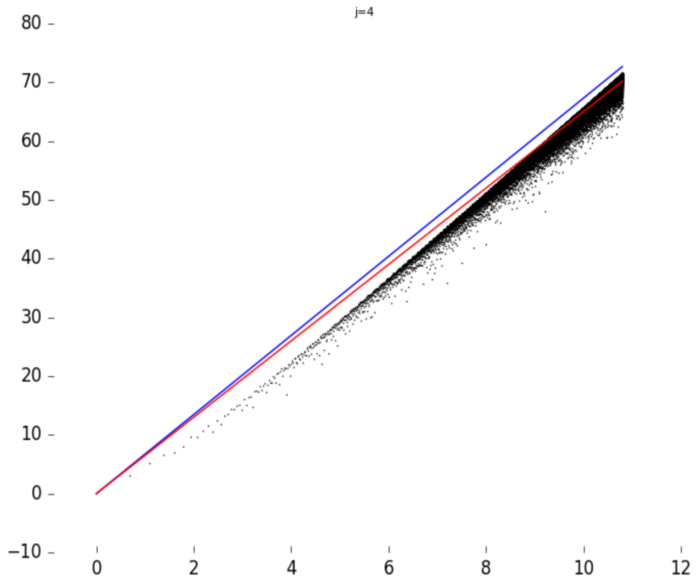
70
60
50
40
30
20
10
0
−10
0
2
4
6
8
10
12


j=3
80
70
60
50
40
30
20
10
0
−10
0
2
4
6
8
10
12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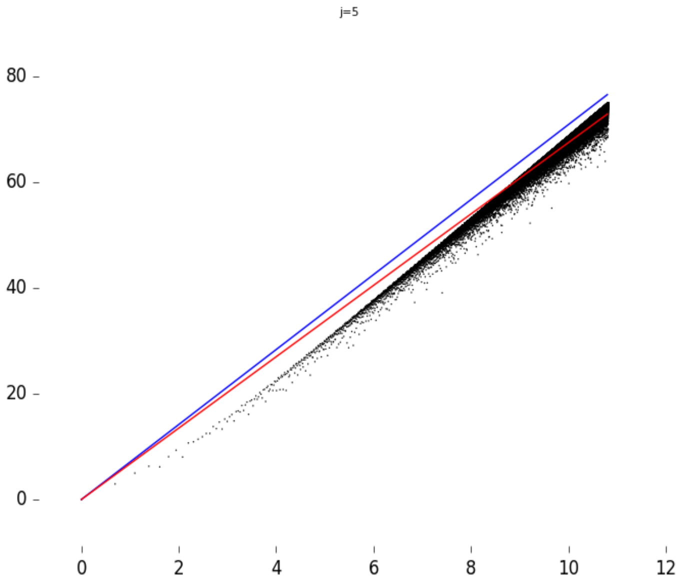
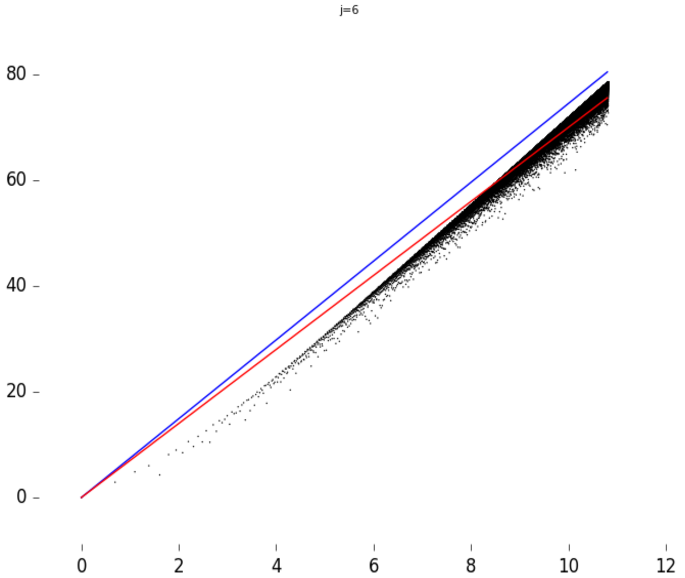
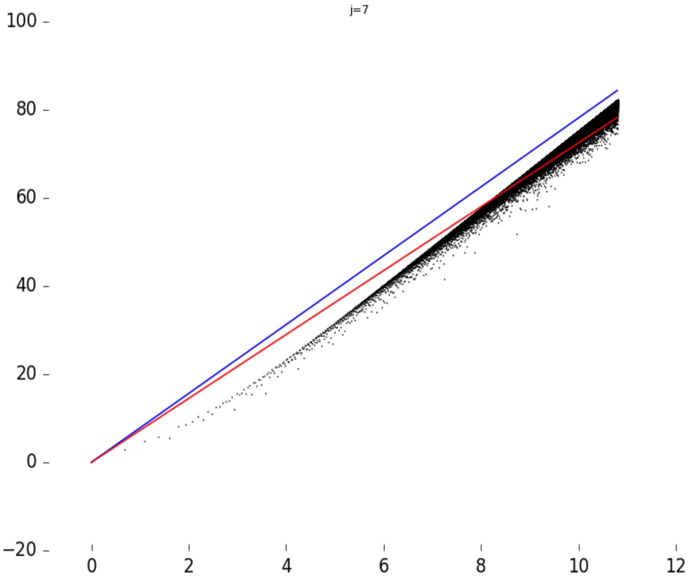
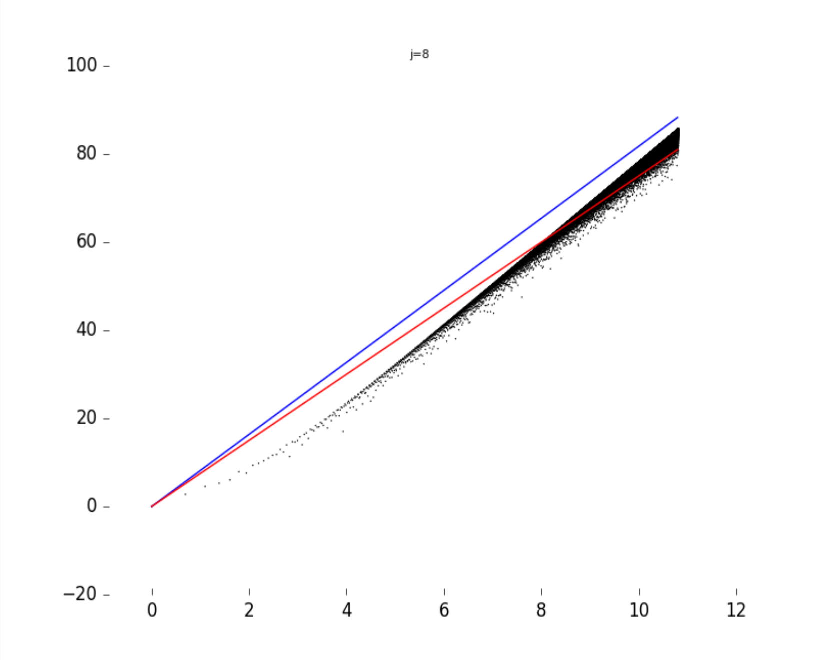
70
60
50
40
30
20
10
0
−10
0
2
4
6
8
10
12

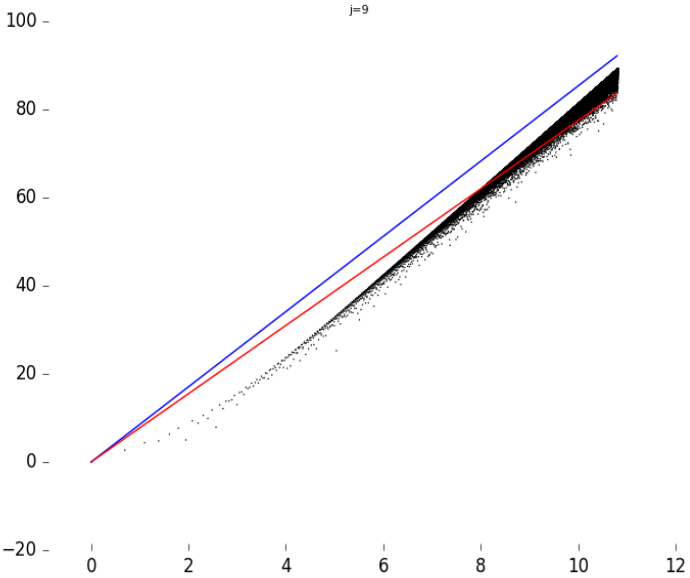

j=5
80
60
40
20
0
0
2
4
6
8
10
12

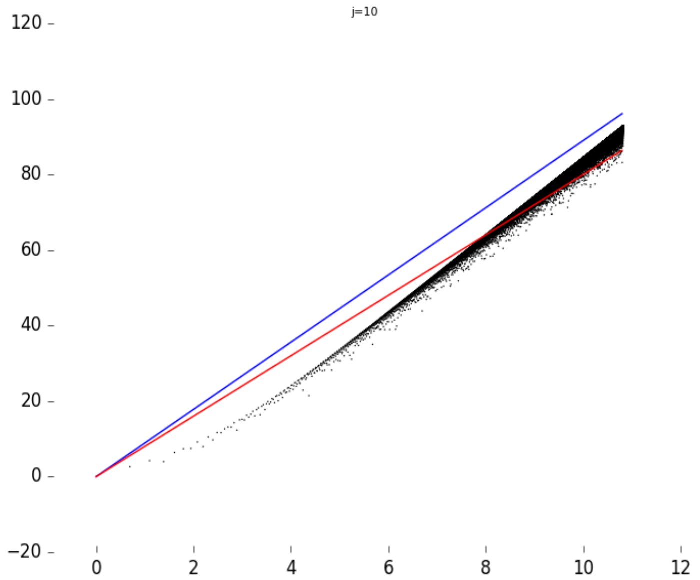

j=6
80
60
40
20
0
0
2
4
6
8
10
12



j=7
100
80
60
40
20
0
−20
0
2
4
6
8
10
12


j=8
100
80
60
40
20
0
−20
0
2
4
6
8
10
12


j=9
100
80
60
40
20
0
−20
0
2
4
6
8
10
12



j=10
120
100
80
60
40
20
0
−20
0
2
4
6
8
10
12

Directions for Further Investigation

Directions for Further Investigation

It's not yet clear what degree of cancellation we should really expect. More numerical experimentation should lead to more precise conjectures, but we conjecture that there is more-than-square-root cancellation in general.

For $S_f^{(j)}(X)$ to be small for j large implies difficult to understand regularity constraints on the sizes and sign changes of the individual coefficients $a(n)$ and smaller iterates. There might be a connection with the Sato-Tate conjecture, but this connection is unexplored.

It is natural to ask about the same question for non-cusp forms, such as those forms leading to the Gauss Circle and Dirichlet Hyperbola methods. We have begun to investigate the approaches mentioned here for these cases.

Questions?

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