A CONDENSED RESTATEMENT OF THE TESTS

DAVID LOWRY-DUDA

1. The nth term test

Suppose we are looking at $\sum_{n=1}^{\infty} a_n$ and

$$\lim_{n\to\infty} a_n \neq 0.$$

Then $\sum_{n=1}^{\infty} a_n$ *does not converge.*

1.1. **Alternating Series Test.**

Suppose $\sum_{n=1}^{\infty} (-1)^n a_n$ is a series where

(1) $a_n \geq 0$,
(2) a_n is decreasing, and
(3) $\lim_{n\to\infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Stated differently, if the terms are alternating sign, decreasing in absolute size, and converging to zero, then the series converges.

2. Geometric Series

Given a geometric series

$$\sum_{n=0}^{\infty} ar^n,$$

the series converges exactly when $|r| < 1$. If $|r| \geq 1$, then the series diverges.

Further, if $|r| < 1$ (so that the series converges), then the series converges to

$$\sum_{n=0}^{\infty} ar^n = \frac{1}{1-r}.$$

3. Telescoping Series

[If a series telescopes, then you can explicitly compute the limit of the partial sums very straightforwardly.]

4. Integral Test

Suppose that $f(x)$ is a positive, decreasing function. Then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ and the integral $\displaystyle\int_1^{\infty} f(x)dx$ either both converge, or both diverge.

5. P-series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

6. Direct comparison

Suppose we are considering the two series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n,$$

where $a_n \geq 0$ and $b_n \geq 0$. Suppose further that

$$a_n \leq b_n$$

for all n (or for all n after some particular N). Then

$$0 \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n.$$

Further, if $\sum_{n=0}^{\infty} a_n$ diverges, then so does $\sum_{n=0}^{\infty} b_n$. And if $\sum_{n=0}^{\infty} b_n$ converges, then so does $\sum_{n=0} \infty a_n$.

This can be restated in the following informal way: if the bigger one converges, then so does the smaller. And in the other direction, if the smaller one diverges, then so does the larger.

7. Limit comparison

Suppose we are considering the series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n,$$

where $a_n \geq 0$ and $b_n \geq 0$. Then if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

and $L \neq 0, \infty$, then the two series either both converge or both diverge.

[Recall that we discussed a stronger version of this statement in class, concerning what can be said when $L = 0$ or $L = \infty$. We don't reinclude that here.]

8. The ratio test

Suppose we are considering

$$\sum_{n=0}^{\infty} a_n.$$

Suppose that the following limit exists:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r.$$

Then if $r < 1$, the series converges absolutely. If $r > 1$, the series diverges.

If $r = 1$, then this test is inconclusive and one must try other techniques.

9. The root test

Suppose that we are considering

$$\sum_{n=0}^{\infty} a_n.$$

If the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$$

exists and $r < 1$, then the series converges absolutely. If the limit exists and $r > 1$, then the series diverges.

If the limit does not exist, or if the limit exists and $r = 1$, then the test is inconclusive and one must try something else.