Continued Fractions and Pell's Equation

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Abstract

Continued fractions provide a useful, and arguably more natural, way to understand and represent real numbers as an alternative to decimal expansions. In this paper, we enumerate some of the most salient qualities of simple continued fraction representations of real numbers, classify the periodic continued fractions as the quadratic irrationals, and use simple continued fractions to find the integer solutions $(x, y) \in \mathbb{Z}^2$ to the generalized Pell's equation $x^2 - Dy^2 = (-1)^m$.

To begin, let us define a finite simple continued fraction. Finite simple continued fractions are a method for representing the rational numbers $\mathbb{Q} = \left\{ \dfrac{a}{b} : a, b \in \mathbb{Z} \, ; \, b \neq 0 \right\}$, and

Definition 1 (Finite simple continued fraction)**.** Let $n \in \mathbb{N}$ be a natural number, and let $(a_i)_{i=0}^n = (a_0, \ldots, a_n)$ be a finite sequence of natural numbers $a_0, \ldots, a_n \in \mathbb{N}$. The *finite simple continued fraction* generated by $(a_i)_{i=0}^n$, denoted $\underset{i=0}{\overset{n}{\mathrm{K}}} \, a_i$, is defined as follows.

$$\underset{i=0}{\overset{n}{\mathrm{K}}} \, a_i = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_n}}}.$$

Now, we use the Euclidean algorithm to generate a finite simple continued fraction representation for any rational number. Recall that the Euclidean algorithm recursively generates sequences $(q_i)_{n=0}^n$ of quotients $q_i \in \mathbb{N}$ and $(r_i)_{n=0}^n$ of remainders $r_i \in \mathbb{N}$ from two integer inputs $a, b \in \mathbb{N}$, and terminates with $r_{n-1} = \gcd(a, b)$ and $r_n = 0$.

Example 1. Let $\dfrac{a}{b} \in \mathbb{Q}$ be a rational number. Let $(q_i)_{i=0}^n$ be the finite sequence of natural numbers $q_1, \ldots, q_n \in \mathbb{N}$ generated by the quotients in the Euclidean algorithm applied to a and b. Then

$$\underset{i=0}{\overset{n}{\mathrm{K}}} \, q_i = \frac{a}{b}.$$

For example, let $a = 38$ and $b = 9$. Note that

$$38 = (4)(9) + 2;$$
$$9 = (4)(2) + 1;$$
$$2 = (2)(2) + 0.$$

So, letting $(a_i)_{i=0}^2 = (4, 4, 2)$, we have that $\dfrac{39}{9} = \overset{2}{\underset{i=0}{\mathrm{K}}}\, a_i = 4 + \dfrac{1}{4 + \dfrac{1}{2}}$.

In this way, we have a canonical way of representing any rational number $\dfrac{a}{b}$ as a finite simple continued fraction. Moreover, it can be shown that every (noninteger) rational number has precisely two finite simple continued fraction representations, each differing only in the last two terms of the finite generating sequences. For example, let $(b_i)_{i=0}^3 = (4, 4, 1, 1)$, and note that

$$\frac{38}{9} = \overset{2}{\underset{i=0}{\mathrm{K}}}\, a_i = 4 + \frac{1}{4 + \dfrac{1}{2}} = 4 + \frac{1}{4 + \dfrac{1}{1 + \dfrac{1}{1}}} = \overset{3}{\underset{i=0}{\mathrm{K}}}\, b_i.$$

Definition 2 (Infinite simple continued fraction)**.** Let $(a_i)_{i=0}^\infty$ be an infinite sequence of natural numbers $a_0, a_1, a_2, \ldots \in \mathbb{N}$. The *infinite simple continued fraction* generated by $(a_i)_{i=0}^\infty$, denoted $\overset{\infty}{\underset{i=0}{\mathrm{K}}}\, a_i$, is defined to be the limit of the partial simple continued fractions.

$$\overset{\infty}{\underset{i=0}{\mathrm{K}}}\, a_i = \lim_{n \to \infty} \overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i. \tag{1}$$

For any infinite sequence $(a_i)_{i=0}^\infty$ of natural numbers $a_i \in \mathbb{N}$, we have defined an infinite sequence of finite simple continued fractions $\left(\overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i \right)_{n=0}^\infty$, where $\overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i$ is called the nth *convergent*. However, it remains to show that such a sequence must converge to a limiting value.

1   Convergence of Simple Continued Fractions

Theorem 1. *Let $(a_i)_{i=0}^\infty$ be an infinite sequence of natural numbers $a_i \in \mathbb{N}$. Then the sequence $\left(\overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i \right)_{n=0}^\infty$ of convergents $\overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i$ converges to a real number $\overset{\infty}{\underset{i=0}{\mathrm{K}}}\, a_i \in \mathbb{R}$.*

Proof. Let p_n and q_n be the numerator and denominator of the nth convergent $\overset{n}{\underset{i=0}{\mathrm{K}}}\, a_i$.

Lemma 1. The infinite sequences $(p_n)_{n=0}^\infty$ and $(q_n)_{n=0}^\infty$ satisfy the following recursion relations.

$$p_n = \begin{cases} a_0, & \text{if } n = 0; \\ a_1 a_0 + 1, & \text{if } n = 1; \text{ and} \\ a_n p_{n-1} + p_{n-2}, & \text{otherwise}, \end{cases} \tag{2}$$

and

$$q_n = \begin{cases} 1, & \text{if } n = 0; \\ a_1, & \text{if } n = 1; \text{ and} \\ a_n q_{n-1} + q_{n-2}, & \text{otherwise}. \end{cases} \tag{3}$$

Proof. By induction on $n \in \mathbb{N}$.

   Base cases: For the first base case, suppose that $n = 0$. Note that

$$\frac{p_0}{q_0} = \underset{i=0}{\overset{0}{\mathrm{K}}} \, a_i = a_i = \frac{a_0}{1},$$

as desired.

   For the second base case, suppose that $n = 1$. Note that

$$\frac{p_1}{q_1} = \underset{i=0}{\overset{1}{\mathrm{K}}} \, a_i = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1},$$

as desired.

   Inductive step: Suppose for the inductive hypothesis that the lemma holds for all $n < k$, for some natural number $k > 1$. Let $(a_i')_{i=0}^{k}$ be the finite sequence of natural numbers $a_i' \in \mathbb{N}$ defined as follows.

$$(a_i')_{i=0}^{k} = \left(a_1, \dots, a_{k-1}, a_k + \frac{1}{a_{k+1}} \right).$$

Note that $\underset{i=0}{\overset{k}{\mathrm{K}}} \, a_i' = \underset{i=0}{\overset{k+1}{\mathrm{K}}} \, a_i$. By the induction hypothesis,

$$\frac{p_{k+1}}{q_{k+1}} = \frac{p_k'}{q_k'} = \frac{a_k' p_{k-1}' + p_{k-2}'}{a_k' q_{k-1}' + q_{k-2}'} = \frac{\left(a_k + \frac{1}{a_{k+1}} \right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}} \right) q_{k-1} + q_{k-2}}$$

$$= \frac{a_{k+1} \left(a_k p_{k-1} + p_{k-2} \right) + p_{k-1}}{a_{k+1} \left(a_k q_{k-1} + q_{k-2} \right) + q_{k-1}} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}},$$

as desired.

$\square$

Lemma 2. $\dfrac{p_{n-1}}{q_{n-1}} - \dfrac{p_n}{q_n} = \dfrac{(-1)^n}{q_{n-1} q_n}$ for all natural numbers $n \in \mathbb{N}$.

Proof. By induction on $n \in \mathbb{N}$.

   Base case: Suppose that $n = 1$. Then $\dfrac{p_0}{q_0} - \dfrac{p_1}{q_1} = \dfrac{a_0}{1} - \dfrac{a_1 a_0 + 1}{a_1} = -\dfrac{1}{a_1} = -\dfrac{1}{q_0 q_1}$, as desired.

   Inductive step: Suppose for the inductive hypothesis that

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-2} q_{n-1}}$$

$$p_{n-2} q_{n-1} - p_{n-1} q_{n-2} = (-1)^{n-1}.$$

Then

$$p_{n-1} q_n - p_n q_{n-1} = p_{n-1} \left(a_n q_{n-1} + q_{n-2} \right) - \left(a_n p_{n-1} + p_{n-2} \right) q_{n-1}$$

$$= p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = (-1)^n$$

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n-1} q_n},$$

as desired.

$\square$

Therefore, since $(q_n)_{n=0}^{\infty}$ is a strictly increasing sequence of integers, for all natural numbers $m > n \in \mathbb{N}$,

$$\left| \underset{i=0}{\overset{m}{\mathrm{K}}} a_i - \underset{i=0}{\overset{n}{\mathrm{K}}} a_i \right| = \left| \frac{p_m}{q_m} - \frac{p_n}{q_n} \right| \leq \sum_{i=n+1}^{m} \left| \frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} \right| = \sum_{i=n+1}^{m} \frac{1}{q_i q_{i-1}} \to 0$$

approaches 0 as $m, n \to \infty$, since q_i grows linearly. Therefore, the convergents $\left(\underset{i=0}{\overset{n}{\mathrm{K}}} a_i \right)_{n=0}^{\infty} =$ $\left(\dfrac{p_n}{q_n} \right)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers $\underset{i=0}{\overset{n}{\mathrm{K}}} a_i \in \mathbb{Q}$, and so converges to some real number $\underset{i=0}{\overset{\infty}{\mathrm{K}}} a_i \in \mathbb{R}$, since the real numbers $\mathbb{R}$ form a complete metric space.

$\square$

2 Continued Fraction Representations of Real Numbers

In many ways, continued fractions are a more natural way to represent real numbers than decimal expansions. As shown above, the generating sequence of the simple continued fraction of the ratio of two numbers is the quotient sequence constructed by the Euclidean algorithm, and so the simple continued fraction representation of a rational number contains a vast wealth of information about the number, whereas the decimal expansions (indeed, the n-ary expansions for any base $n \in \mathbb{N}$) of many simple fractions obscure such information.

The Euclidean algorithm technique demonstrated above can be generalized to find the continued fraction representation of an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ in the same manner. However, such an algorithm is not guaranteed to terminate as it is in the case of the rational numbers $\mathbb{Q}$. In fact, if a number $x \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, such a process cannot possibly terminate, since termination would imply that there is a finite simple continued fraction representation for x, which would imply that x were rational.

As shown below, the simple continued fraction representations of some irrational numbers have interesting and beautiful forms. We will see an example of such beauty, and then we will see a theorem about those irrational numbers whose simple continued fraction representations repeat.

Example 2. Let $\varphi = \dfrac{1 + \sqrt{5}}{2}$ be the golden ratio. φ has continued fraction representation

$$\varphi = \underset{i=0}{\overset{\infty}{\mathrm{K}}} 1 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}} \tag{4}$$

Proof. Consider the simple continued fraction $\underset{i=0}{\overset{\infty}{\mathrm{K}}} 1$. Since it repeats with a period of 1, we have the simple recurrence relation $\underset{i=0}{\overset{\infty}{\mathrm{K}}} = 1 + \dfrac{1}{\mathrm{K}_{i=0}^{\infty}}$, which follows immediately from Equation (4). This

implies that

$$0 = \left(\mathop{\mathrm{K}}_{i=0}^{\infty} 1 \right)^2 - \mathop{\mathrm{K}}_{i=0}^{\infty} -1$$

$$\mathop{\mathrm{K}}_{i=0}^{\infty} = \frac{1 \pm \sqrt{5}}{2}.$$

From our definition of $\mathop{\mathrm{K}}\limits_{i=0}^{\infty} 1$ in Equation (4), we know that $\mathop{\mathrm{K}}\limits_{i=0}^{\infty} 1 > 1$, so $\mathop{\mathrm{K}}\limits_{i=0}^{\infty} 1 = \dfrac{1 + \sqrt{5}}{2} = \varphi$, as desired.

$\square$

The above technique may be generalized and used to show that any repeating simple continued fraction is an irrational solution to a quadratic equation with integer coefficients, by solving the recurrence relation generated by the repetition. We offer the following theorem without proof.

Theorem 2. *Let $(a_i)_{i=1}^{\infty}$ be an infinite sequence of natural numbers $a_i \in \mathbb{N}$. If $(a_i)_{i=1}^{\infty}$ eventually repeats, then $\alpha = \mathop{\mathrm{K}}\limits_{i=0}^{\infty} a_i$ is an irrational solution to a quadratic equation with integer coefficients.*

Proof. Suppose that $(a_i)_{i=1}^{\infty}$ eventually repeats. It suffices to show that $\alpha = \mathop{\mathrm{K}}\limits_{i=1}^{\infty} a_i$ is a quadratic irrational in the case where $(a_i)_{i=1}^{\infty}$ is purely periodic. Therefore, let $(a_i)_{i=1}^{\infty} = \overline{(a_0, \ldots, a_n)}$. So

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\alpha}}}}.$$

We may reduce this to $\alpha = \dfrac{u\alpha + v}{w\alpha + z}$, for some positive integers $u, v, w, z \in \mathbb{N}$. So

$$\alpha \left(w\alpha + z \right) = u\alpha + v$$
$$w\alpha^2 + z\alpha = u\alpha + v$$
$$w\alpha^2 + (z - u)\alpha - v = 0.$$

So α is a solution to the quadratic equation $wx^2 + (z - u)x - v = 0$. Moreover, since $\alpha = \mathop{\mathrm{K}}\limits_{i=0}^{\infty} a_i$ is an infinite simple continued fraction, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is irrational. So α is a quadratic irrational, as desired.

$\square$

3 Pell's Equation

Let $D \in \mathbb{N}$ be a natural number, and suppose that D is not a perfect square; that is, there does not exist an integer $k \in \mathbb{Z}$ such that $k^2 = D$. The Diophantine equation

$$x^2 - Dy^2 = 1 \tag{5}$$

is known as Pell's equation, and we can use the simple continued fraction representation of $\sqrt{D}$ to solve it.

Theorem 3. *Let $D \in \mathbb{N}$ be a natural number, and suppose that D is not a perfect square. $\sqrt{D}$ has simple continued fraction representation generated by the repeating sequence $(a_0, \overline{a_1, \ldots, a_m})$. Let $\dfrac{p}{q} = \overset{m-1}{\underset{i=0}{\mathrm{K}}} a_i$. Then $(x, y) = (p, q) \in \mathbb{Z}^2$ are the smallest integer solutions to the generalized Pell's equation,*

$$x^2 - Dy^2 = (-1)^m. \tag{6}$$

Example 3. Consider Pell's equation in the case $D = 21$,

$$x^2 - 21y^2 = 1. \tag{7}$$

Note that $\sqrt{21}$ has continued fraction representation $\sqrt{21} = \mathrm{K}\left(4, \overline{1, 1, 2, 1, 1, 8}\right)$, so $m = 6$. $m - 1 = 5$, and the 5th convergent is

$$\mathrm{K}\left(4, 1, 1, 2, 1, 1\right) = 4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}}} = \frac{55}{12}.$$

So $(x, y) = (55, 12)$ is the smallest solution in integers to the generalized Pell's equation, Equation (7), $x^2 - 21y^2 = (-1)^m = 1$, as desired. Moreover, the solution set S to Pell's equation is as follows.

$$S = \left\{ (x_k, y_k) \in \mathbb{Z}^2 : \left(55 + 12\sqrt{21}\right)^k = x_k + y_k\sqrt{21} \right\}_{k=1}^{\infty}.$$

Example 4. We may still solve Pell's equation in the case where m is odd, by squaring the solutions. Consider Pell's equation in the case $D = 29$,

$$x^2 - 29y^2 = 1. \tag{8}$$

Note that $\sqrt{29}$ has continued fraction representation $\sqrt{29} = \mathrm{K}\left(5, \overline{2, 1, 1, 2, 10}\right)$, so $m = 5$. $m - 1 = 4$, and the 4th convergent is

$$K(5, 2, 1, 1, 2) = 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} = \frac{70}{13}.$$

So $(x, y) = (70, 13)$ is the smallest solution in integers to the generalized Pell's equation, $x^2 - 29y^2 = (-1)^m = -1$, as desired. So, squaring both sides of the generalized Pell's equation, the smallest solution in integers to Pell's equation, Equation (8), is $(x, y) = \left(70^2 + 29\left(13\right)^2, 2\left(70\right)\left(13\right)\right) = (9801, 1820)$. Moreover, the solution set S to Pell's equation is as follows.

$$S = \left\{ (x_k, y_k) \in \mathbb{Z}^2 : \left(9801 + 1820\sqrt{29}\right)^k = x_k + y_k\sqrt{21} \right\}_{k=1}^{\infty}.$$

Continued fractions provide a natural and useful way to approach representing real numbers. The generating sequences of simple continued fractions are interestingly connected to the Euclidean algorithm, which stems from the natural association of rational numbers and greatest common divisors. Moreover, the simple continued fraction representations of the quadratic irrationals provide a quick and easy way to solve Pell's equation, which would otherwise prove difficult.

References

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