Introduction to Number Theory

Spring 2016

Homework # 8
Last Updated: April 7, 2016
Due Date: Thursday April 14th.

I recommend that you read Chapters 35 and 36 on $\mathbb{C}$ and $\mathbb{Z}[i]$, even though our point of view is different from the point of view in the textbook.

FRINT Chapter 35:

(1) 35.3
(2) 35.4
(3) Let $N(z)$ denote the *norm* of z, as we've called it in class. Show that $z\overline{z} = N(z)$.
(4) Show that $N(zw) = N(z)N(w)$.

I encourage you to read the proofs of Theorems 36.2, 36.3, and 36.4 from the textbook. These offer another proof of unique factorization in $\mathbb{Z}[i]$, but which is different than the methodology we followed in class.

FRINT Chapter 36:

(5) 36.2
(6) 36.4 (This exercise is easier if you use what we've shown in class about the Euclidean Algorithm for $\mathbb{Z}[i]$ instead of following the hints in the book)

In the next exercise, you will investigate some properties of the ring of integers $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. For $n = a + b\sqrt{-5}$, we define $\overline{n} = a - b\sqrt{-5}$ to be the *conjugate* of n, and $N(n) = a^2 + 5b^2 = n\overline{n}$ to be the *norm* of n.

(7) Show that $3 + 2\sqrt{-5}$ divides $85 - 11\sqrt{-5}$.
(8) Show that $N(ab) = N(a)N(b)$.
(9) The number 6 can be factored as $2 \cdot 3$, which are are familiar with, but also as $(1 + \sqrt{-5})(1 - \sqrt{-5})$, which may feel a bit more exotic. Show that 2 does not divide either $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$.
(10) Does $\mathbb{Z}[\sqrt{-5}]$ have unique factorization?