Homework #2 Solutions

$\boxed{5.1}$ a, b, c, d

Pythagorean Triples have the form $(a,b,c) = (u^2-v^2, 2uv, u^2+v^2)$.

(a) If $\gcd(u,v) = g > 1$, then $g \mid u^2-v^2, 2uv, u^2+v^2$ (and in fact $g^2 \mid u^2-v^2, 2uv, u^2+v^2$), so that the triple has the common factor g. ■

(b) $(u,v) = (3,1)$ has $(u^2-v^2, 2uv, u^2+v^2) = (8, 6, 10)$, a multiple of a $(3,4,5)$ triangle. ■

(further, choosing many such (u,v) are possible.)

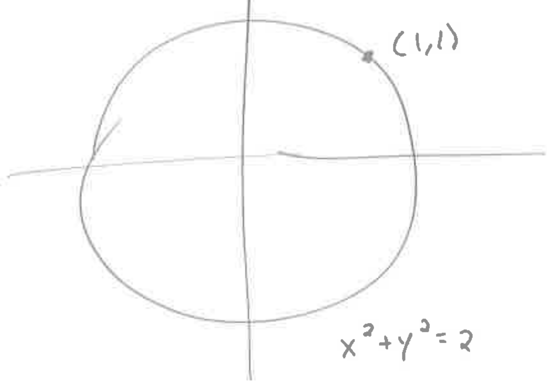
(c) [Make a table]

(d) Some nice conditions are that

① $\gcd(u,v) = 1$.

② Exactly one of u or v is even, + the other is odd.

3.2



A line through $(1,1)$ has equation
$$y = m(x-1)+1 = mx+(1-m),$$
where m is the slope.
The circle is given by $x^2+y^2=2$.

To find intersections, we find simultaneous solutions
$$\begin{cases} x^2+y^2=2 \\ y = mx+(1-m) \end{cases}$$

Let's substitute $y = mx+(1-m)$ into the equation for the circle.
$$x^2 + (mx+(1-m))^2 = 2,$$ which simplifies to
$$(m^2+1)x^2 - 2(m^2-m)x + (m^2-2m-1) = 0.$$
As $x=1$ is a solution, we can factor out $(x-1)$ to get that
$$(m^2+1)x^2 - 2(m^2-m)x + (m^2-2m-1) =$$
$$= (x-1)\left[(m^2+1)x - (m^2-2m-1)\right].$$
So the other root is $x = \dfrac{m^2-2m-1}{m^2+1}$.

The corresponding y-coordinate is
$$y = mx+(1-m)$$
$$= m\,\frac{m^2-2m-1}{m^2+1} + (1-m)$$
$$= \frac{-m^2-2m+1}{m^2+1}.$$

So rational points on $x^2+y^2=2$ are those points coming from rational m, of the form

$$(x,y) = \left(\frac{m^2-2m-1}{m^2+1} , \frac{-m^2-2m+1}{m^2+1} \right). \blacksquare$$

(b) $x^2+y^2=3$ doesn't have any rational points at all, and we need a point to start this process. $\blacksquare$

3.5
(a) We did this in class, but let's remind ourselves.

n^{th} Triangular Number: $\frac{n(n+1)}{2}$

m^{th} Square Number: m^2

So we want $m^2 = \frac{n(n+1)}{2}$, or equivalently

$$8m^2 = 4n(n+1) = 4n^2+4n+1-1$$
$$= (4n^2+4n+1)-1$$
$$= (2n+1)^2-1.$$

Call $x=2n+1$, $y=2m$.

Then $2(2m)^2 = (2n+1)^2-1$ is the same as

$2y^2 = x^2-1$, which is our hyperbola.

We want solutions where y is even and x is odd. $\blacksquare$

(b) $x^2 - 2y^2 = 1$

The line through $(1,0)$ with slope m has equation

$$y = m(x-1).$$

Substituting into $x^2 - 2y^2 = 1$ and solving, we find the other point

$$(x,y) = \left(\frac{2m^2+1}{2m^2-1}, \frac{2m}{2m^2-1} \right). \quad \blacksquare$$

(c) Writing $m = \frac{v}{u}$, we rewrite (x,y) as

$$\left(\frac{2\frac{v^2}{u^2}+1}{2\frac{v^2}{u^2}-1}, \frac{2\frac{v}{u}}{2\frac{v^2}{u^2}-1} \right),$$ which after multiplying by $\frac{u^2}{u^2}$ becomes

$$\rightsquigarrow \left(\frac{2v^2+u^2}{2v^2-u^2}, \frac{2vu}{2v^2-u^2} \right).$$

If $u^2 - 2v^2 = 1$, the denominators are -1, so that the other point (after changing signs) is

$$(2v^2+u^2, 2vu). \quad \blacksquare$$

(d) Starting with $(3,2)$, the next one from (b)+(c) is $(2\cdot 2^2 + 3^2, 2\cdot 2\cdot 3) = (17,12)$. Starting with $(17,12)$ gives $(577, 408)$. Then $(665857, 470832)$.

To get square-triangular numbers from these, we need to set $2n+1 = x$, $2m = y$. Or rather, $n = \frac{x-1}{2}$, $m = \frac{y}{2}$.

Then these values correspond to

$$(3,2) \longrightarrow \left(\frac{3-1}{2}, \frac{2}{2}\right) = (1,1), \text{ where } m^2 = 1.$$

$$(17,12) \longrightarrow \left(\frac{16}{2}, \frac{12}{2}\right) = (8,6), \text{ where } m^2 = 36.$$

$$(577,408) \longrightarrow (288, 204), \text{ where } m^2 = (204)^2 = 41616.$$

$$(665857, 470832) \longrightarrow (332928, 235416), \text{ where}$$

$$m^2 = (235416)^2 = 5\,542\,069\,3056.$$

{I include the 4th to show that this does get us further than we could have gotten on the 1st homework.}

(e) Starting with solution (u,v), the new y-coordinate is $2uv$. This is always larger than v, so the y-coordinates are always increasing. Thus each time we get a new solution. ■

[Note: I know this was a challenging problem. But I think it's so nice of an example of how lines and geometry can help us towards otherwise extremely challenging + impossible problems.]

5.1

(a)

$\gcd(12345, 67890) = 15.$

$67890 = 5 \cdot 12345 + 6165$

$12345 = 2 \cdot 6165 + 15$

$6165 = 411 \cdot 15 + 0$

(b)

$\gcd(54321, 9876) = 3.$

$54321 = 5 \cdot 9876 + 4941$

$9876 = 1 \cdot 4941 + 4935$

$4941 = 1 \cdot 4935 + 6$

$4935 = 822 \cdot 6 + 3$

$6 = 2 \cdot 3 + 0$

5.4

$LCM(8,12) = 24$	$\gcd(8,12) = 4$	$8 \cdot 12 = 96$
$LCM(20,30) = 60$	$\gcd(20,30) = 10$	$20 \cdot 30 = 600$
$LCM(51,68) = 204$	$\gcd(51,68) = 17$	$51 \cdot 68 = 3468$
$LCM(23,18) = 414$	$\gcd(23,18) = 1$	$23 \cdot 18 = 414$

The relationship is $\gcd(a,b) \cdot LCM(a,b) = a \cdot b.$
Did you know this already?

5.5

(a) $21 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$ length 8

$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ length 10

For 31 ... this was a bit cruel. The length is 107.
Isn't it surprising how large these numbers can get?
The next number with longer length is 41, of length 110.

(b) What do you think? Many think it always terminates in a $4 \to 2 \to 1$ loop. We don't know, though.

(c) Notice $8k+4 \to 4k+2 \to 2k+1 \to 6k+4$, so it

$8k+5 \to 24k+16 \to 12k+8 \to 6k+4$

takes 3 steps before they fall into the same sequence.

$\boxed{\#7}$

The only certainty is that the steps must look something like

$$
\begin{aligned}
\underline{\qquad} &= \underline{\qquad} \cdot \underline{A} + 9 \\
\underline{A} &= \underline{\qquad} \cdot 9 + 4 \\
9 &= 2 \cdot 4 + 1 \\
4 &= 1 \cdot 4 + 0
\end{aligned}
$$

where the blanks are open, and the "A" spots are the same (and $A > 9$). One possibility is

$$
\begin{aligned}
58 &= 1 \cdot 49 + 9 \\
49 &= 5 \cdot 9 + 4 \\
9 &= 2 \cdot 4 + 1 \\
4 &= 4 \cdot 1 + 0
\end{aligned}
$$

another is

$$
\begin{aligned}
197 &= 2 \cdot 94 + 9 \\
94 &= 10 \cdot 9 + 4 \\
9 &= 2 \cdot 4 + 1 \\
4 &= 4 \cdot 1 + 0
\end{aligned}
$$

, but there are infinitely many. ∎