CONTINUITY OF THE MEAN VALUE

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1. Introduction

When I first learned the mean value theorem as a high schooler, I was thoroughly unimpressed. Part of this was because it's just like Rolle's Theorem, which feels obvious. But I think the greater part is because I thought it was useless. And I continued to think it was useless until I began my first proof-oriented treatment of calculus as a second year at Georgia Tech. Somehow, in the interceding years, I learned to value intuition and simple statements.

I have since completely changed my view on the mean value theorem. I now consider essentially all of one variable calculus to be the Mean Value Theorem, perhaps in various forms or disguises. In my earlier note An Intuitive Introduction to Calculus (see [1]), we state and prove the Mean Value Theorem, and then show that we can prove the Fundamental Theorem of Calculus with the Mean Value Theorem and the Intermediate Value Theorem (which also felt silly to me as a high schooler, but which is not silly).

In this brief note, I want to consider one small aspect of the Mean Value Theorem: can the "mean value" be chosen continuously as a function of the endpoints? To state this more clearly, first recall the theorem:

Theorem 1.1 (Mean Value Theorem)**.** *Suppose f is a differentiable real-valued function on an interval $[a, b]$. Then there exists a point c between a and b such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c), \tag{1.1}$$

which is to say that there is a point where the slope of f is the same as the average slope from a to b.

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What if we allow the interval to vary? Suppose we are interested in a differentiable function f on intervals of the form $[0, b]$, and we let b vary. Then for each choice of b, the mean value theorem tells us that there exists c_b such that

$$\frac{f(b) - f(0)}{b} = f'(c_b).$$

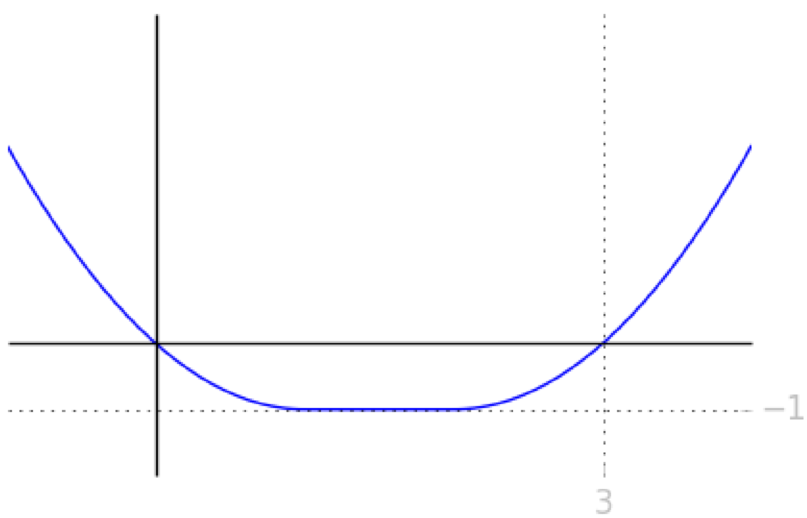
Then the question we consider today is, as a function of b, can c_b be chosen continuously? We will see that we cannot, and we'll see explicit counterexamples.

2. A Counterexample

For ease, we will restrict ourselves to intervals of the form $[0, b]$, as mentioned above. A particularly easy counterexample is given by

$$f(x) = \begin{cases} x^2 - 2x & x \leq 1 \\ -1 & -1 \leq x \leq 2 \\ x^2 - 4x + 3 & x \geq 2 \end{cases}$$

This is a flattened parabola, that is, a parabola with a flattened middle section.



Clearly, the slope of the function f is negative until $x = 1$, where it is 0. It becomes (and stays) positive at $x = 2$. So if you consider intervals $[0, b]$ as b is varying, since $f(b) < 0$ for $x < 3$, we must have that c_b is at a point when $f'(c_b) < 0$, meaning that $c_b \in [0, 1]$. But as soon as $b > 3$, $f(b) > 0$ and c_b must be a point with $f'(c_b) > 0$, meaning that $c_b \in [2, b]$.

In particular, c_b jumps from at most 1 to at least 2 as b goes to the left of 3 to the right of 3. So there is no way to choose the c_b values locally in a neighborhood of 3 to make the mean values continuous there.

Further, although this example is not smooth, it is easy to see that if we "smoothed" off the connection between the parabola and the straight line, like through the use of bump functions, then the spirit of this counterexample works, and not even smooth functions have locally continuous choices of the mean value.

This short note also appears on the authors main website [2], and has a lovely gif too.

References

[1] David Lowry-Duda, *An Intuitive Introduction to Calculus*, `http://davidlowryduda.com/?p=1259`, author's personal website, September 2013.

[2] David Lowry-Duda, MixedMath, `http://davidlowryduda.com/`, author's personal website.