Math 100 Week 6 Recitation

7 November 2013

1) (a) $\frac{1}{e}, \quad \frac{2^4}{e^2}, \quad \frac{3^4}{e^3}, \quad \frac{4^4}{e^4} \longrightarrow 0$ as exponentials beat polynomials

(b) $\frac{3}{2+100000}, \quad \frac{9}{4+100000}, \quad \frac{27}{8+100000}, \quad \longrightarrow \infty$

for large n, looks like $\frac{3^n}{2^n} \to \infty$.

(c) $\frac{1}{2!}, \quad \frac{3}{4!}, \quad \frac{5\cdot 3}{6!} \longrightarrow 0$ as each increase in n by 1 multiplies top by $\approx n$ + bottom by $\approx n^3$;

i.e. "ratio test" $\rightsquigarrow \frac{n}{n^3} \longrightarrow 0$.

2) $a_n = 6\cdot\left(\frac{1}{10}\right)^{n} \cdot (-1)^n = 6\cdot(-10)^n$, $n \geq 0$. Limit is 0. (geometric

$b_n = \left(5 - \frac{1}{n}\right)^{1/n}$ for $n \geq 1$. $\lim \approx 5^{1/n} \longrightarrow 1.$

$$\lim_{n\to\infty} \left(5 - \frac{1}{n}\right)^{1/n} = L$$

long, real way to do it.

I illustrative, I think.

$$\Rightarrow \lim \frac{1}{n}\log\left(5 - \frac{1}{n}\right) = \log L$$

$$\Rightarrow \lim \frac{\log\left(5-\frac{1}{n}\right)}{n} \overset{\text{L'Hôpital}}{=} \lim \frac{\frac{1}{5-\frac{1}{n}}\cdot\frac{1}{n^2}}{1} = \lim \frac{1}{5n^2 - \frac{1}{n^3}} \longrightarrow 0 = \log L$$

$$\Rightarrow L = e^0 = 1$$

$c_n = \frac{n^2}{2n^2+4n+3} \longrightarrow \frac{1}{2}$

3) $a_1 = 9$, $a_{n+1} = 15 - a_n$

$9, 6, 9, 6, 9, \ldots$ no limit

$b_1 = 10$, $b_{n+1} = 20 - a_n$

$10, 10, 10, 10 \longrightarrow 10$

$c_1 = 11$, $c_{n+1} = \sqrt{c_n + 5}$

$11, 4, 3, \sqrt{8}, \ldots$

c_n has a limit, but it's difficult to verify. (don't worry, it's weird).

In short, $L = \sqrt{L+5} \Rightarrow L^2 = L+5$

$\Rightarrow L^2 - L - 5 = 0$

$\Rightarrow L = \frac{1 \pm \sqrt{1+20}}{2} = \boxed{\frac{1+\sqrt{21}}{2}}$

$\boxed{4}$ (Challenge Problem)

$f(1) = 0$ and $g(1) = 0$

$\int_a^b \sqrt{4x^4 + 4x^2 + 2}\, dx$ is the arclength

i.e. WT Find f s.t. $(f')^2 + 1 = 4x^4 + 4x^2 + 2$

and g s.t. $(g')^2 + 1 = 4x^4 + 4x^2 + 2$.

$$\begin{cases} \Longrightarrow (f')^2 = 4x^4 + 4x^2 + 1 = (2x^2+1)^2 \\ \Longrightarrow (g')^2 = (2x^2+1)^2 \end{cases}$$

$$\begin{cases} \Rightarrow f' = \pm(2x^2+1) & \text{say} = 2x^2+1 \\ g' = \pm(2x^2+1) & \text{say} = -2x^2-1 \end{cases}$$

$\Rightarrow f = \cancel{\int 2y^2 + 1\, dy}\ \int 2x^2+1 = \frac{2x^3}{3} + x + A$

want $f(1) = 0 \Rightarrow \frac{2}{3} + 1 = -A$

$\Rightarrow A = -\frac{5}{3}$

$\Rightarrow f(x) = \frac{2}{3}x^3 + x - \frac{5}{3}.$

$\Rightarrow g = -\int 2x^2 + 1\, dx = -\frac{2}{3}x^3 - x + B$

want $g(1) = 0 \Rightarrow B = \frac{2}{3} + 1 = \frac{5}{3}$

$\Rightarrow g(x) = -\frac{2}{3}x^3 + -x + \frac{5}{3}$

So $\pm\left(\frac{2}{3}x^3 + x - \frac{5}{3}\right)$ are the only 2 such functions.