Math 100 Week 10 Recitation

7 November 2013

$\boxed{1}$ (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$ A geometric series with first term $\frac{-2}{3}$ and ratio $r = -\frac{2}{3}$. $|\text{Ratio}|$ is $< 1 \Rightarrow$ converges.

The sum starts $\frac{-2}{3} + \frac{4}{9} + -\frac{8}{27}$.

It's alternating, so the error from the 1st 3 terms is less than the next (4th in this case) term. So the error is less than $\frac{16}{81}$.

Aside: This is geometric, so we actually know the exact sum: $\frac{-2/3}{1-(-2/3)}$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$ converges by alternating series test.

The sum starts $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$.

It's alternating, so the error from the first 3 terms is less than the next term,

So the error is less than $\frac{1}{5!} = \frac{1}{120}$, which is pretty good.

This converges very quickly.

$\boxed{2}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(x-6)^n}{2^n n!}$ (a) $x=5 \rightsquigarrow \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

$\frac{1}{2^n n!} < \frac{1}{500} \rightsquigarrow$ solve for n. $\Longleftrightarrow 2^n n! > 500$

$2^4 4! = 16 \cdot 24 < 500$

$2^5 \cdot 5! = 32 \cdot 120 > 500$

So $\underbrace{1 + \frac{-1}{2} + \frac{1}{4 \cdot 2!} - \frac{1}{2^3 3!} + \frac{1}{2^4 4!}}_{5 \text{ terms.}}$ is within $\frac{1}{500}$ of the actual.

(b) $x = 5.9 \rightsquigarrow \displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{-1}{10}\right)^n}{2^n n!}$ $\quad \frac{1}{10^n 2^n n!} < \frac{1}{500} \Longleftrightarrow 10^n 2^n n! > 500$

$\Longleftrightarrow 20^n n! > 500$

$n=2 \rightsquigarrow 20^2 \cdot 2 = 800 > 500$

So $\underbrace{1 - \frac{1}{10 \cdot 2 \cdot 1}}_{2 \text{ terms}}$ is within $\frac{1}{500}$ of the actual sum.

Aside: (b) converges faster because x is closer to the center of its interval of convergence.

(c) b converges (much) faster.

3. $\sum_{n=0}^{\infty} \frac{(2x+5)^{n+1} n^{10}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{2^{n+1}(x+\frac{5}{2})^{n+1} n^{10}}{(n+1)!}$

(a) So it's centered around $a = -\frac{5}{2}$ (viewable), or by seeing that it's trivially 0 at $x = -\frac{5}{2}$, + power series are always trivial at their center).

(b) Ratio test: $\lim_{n\to\infty} \left| \frac{(2x+5)^{n+2} (n+1)^{10}}{(n+2)!} \cdot \frac{\cancel{(n+1)!}}{\cancel{(2x+5)^{n+1}} n^{10}} \right| = \lim_{n\to\infty} \frac{(2x+5)}{n+2} \frac{(n+1)^{10}}{n^{10}}$ ← degree 10 ← degree 11

$(n+2)! = (n+2)\cancel{(n+1)!}$

$\longrightarrow 0$

So the interval of convergence is $\mathbb{R}$, or $(-\infty, \infty)$.

(this is because factorials are very, very large).

4. $f(x) = \sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n^2+3}$

(a) Root test: $\sqrt[n]{\left|\frac{3^n (x-2)^n}{n^2+3}\right|} = |3(x-2)| < 1$

$\Rightarrow |x-2| < \frac{1}{3} \Rightarrow -\frac{7}{3} < x < -\frac{5}{3}$

what about endpoints?

@ $x = -\frac{7}{3}$, $\rightsquigarrow \sum \frac{3^n(-\frac{1}{3})^n}{n^2+3} = \sum \frac{(-1)^n}{n^2+3}$ converges by alternating series test

@ $x = -\frac{5}{3}$ $\rightsquigarrow \sum \frac{3^n(\frac{1}{3})^n}{n^2+3} = \sum \frac{1^n}{n^2+3} < \sum \frac{1}{n^2}$ converges by p series, $p = 2 > 1$.

basic comparison

$\Rightarrow$ interval of convergence is $\boxed{[-\frac{7}{3}, -\frac{5}{3}]}$

Aside: the radius of convergence of f will always be the same as the radius of f'. Only differences can be endpoints.

(b) $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{3^n(x-2)^n}{n^2+3} = \sum_{n=1}^{\infty} \frac{3^n n (x-2)^{n-1}}{n^2+3}$

Note that this lower limit changed!

(c) Root test $\rightsquigarrow$ again $|3(x-2)| < 1$

$\Rightarrow -\frac{7}{3} < x < -\frac{5}{3}$.

Endpoints? @ $x = -\frac{7}{3}$ $\rightsquigarrow \sum \frac{3^n(-\frac{1}{3})^n n}{n^2+3} = \sum \frac{(-1)^n n}{n^2+3}$ converges by alternating series test.

@ $x = -5/3$ $\rightsquigarrow \sum \frac{3^n(\frac{1}{3})^n n}{n^2+3} = \sum \frac{n}{n^2+3} \sim \sum \frac{1}{n}$ diverges by p series.

$\lim_{n\to\infty} \frac{n}{n^2+3} \cdot \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+3} = 1$

limit comparison

so $\boxed{[-\frac{7}{3}, -\frac{5}{3})}$

$\boxed{5}$ $f(x) = \ln(2x+1)$

$f'(x) = 2 \cdot (2x+1)^{-1}$

$f''(x) = -1 \cdot 2^2 \cdot (2x+1)^{-2}$

$f'''(x) = -2 \cdot -1 \cdot 2^3 (2x+1)^{-3}$

$f^{(n)}(x) = (-n+1) \cdot \cdots \cdot (-2)\cdot(-1)\cdot 2^n (2x+1)^{-n}$

(a) $\Rightarrow f^{(n)}(4) = (-n+1) \cdot \cdots \cdot (-2)(-1)\, 2^n (9)^{-n} = (-1)^{n-1}(n-1)!\, 2^n 9^{-n}$

(b) $\Rightarrow f(x) = f(4) + \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\, 2^n 9^{-n} \frac{(x-4)^n}{n!} = \ln(9) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n\, 9^n}(x-4)^n$

(c) $f'(x) = \frac{d}{dx}\left(\ln(9) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n\,9^n}(x-4)^n \right)$

$= 0 + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^n}{9^n}(x-4)^{n-1} = \frac{d}{dx}\ln(2x+1) = \frac{2}{2x+1}$

should

Aside: (2nd method, which I don't expect of you).

We know $\frac{d}{dx}\ln(2x+1) = \frac{2}{2x+1}$. But we also know that

$$\frac{2}{1+2x} = \frac{2}{1+2(x-4)+8} = \frac{2}{9+2(x-4)} = \frac{2}{9}\underbrace{\left(\frac{1}{1-\frac{-2}{9}(x-4)}\right)}_{\text{of the form } \frac{1}{1-y}}$$

& $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$ geometric series.

$\Rightarrow \frac{2}{1+2x} = \frac{2}{9}\left(\frac{1}{1-\left(-\frac{2}{9}(x-4)\right)}\right) = \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{-2}{9}\right)^n (x-4)^n$ (which $= \sum_{n=0}^{\infty} (-1)^{n-1}\frac{2^n}{9^n}(x-4)^{n-1}$ that we found above)

$\Rightarrow \ln(2x+1) = \int_4^x \frac{2}{1+2y}\,dy = \int_4^x \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{-2}{9}\right)^n (y-4)^n\,dy =$

$= \frac{2}{9}\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{9}\right)^n (x-4)^{n+1}}{n+1} + \ln 9$ (which $= \ln 9 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n\,9^n}(x-4)^n$ that we found above)

But this was done without taking lots of derivatives!