Brown University - Math 0090

Midterm Exam 1 - October 10, 2012

NAME: Lowry's Key

SECTION (Number and Professor):

Do not open the exam until you are told to do so. Please write your name and read the instructions on this page until you are instructed to begin.

You will have ninety minutes to complete the exam. There are eight problems, some of which have multiple parts. Use your time wisely; if you find yourself stuck on any problems, make sure you attempt all of the other problems before you run out of time.

You must show your work. Correct answers with no explanation may not be given any credit, while incorrect or incomplete credits where some (correct) work is shown are likely to receive some partial credit.

You may not refer to external sources such as notes, your textbook or a calculator. If you wish, you may use the backs of the pages for scratchwork. Good luck!

1	2	3	4	5	6	7	8	TOTAL (of 80)

1. Determine the derivative of each of the following functions.

(a)

$$f(x) = x^4 + 2e^x$$

$f'(x) = 4x^3 + 2e^x$

(b)

$$g(x) = \frac{x^{7/2} - 5}{x^2}$$

Product Rule: $g(x) = (x^{7/2} - 5)\, x^{-2} \longrightarrow g'(x) = \frac{7}{2} x^{5/2} x^{-2} + (x^{7/2} - 5)(-2)x^{-3}$

$= \boxed{\frac{7}{2} x^{1/2} + \frac{-2(x^{7/2} - 5)}{x^3}}$

Quotient Rule: $g'(x) = \boxed{\frac{x^2\left(\frac{7}{2} x^{5/2}\right) - (x^{7/2} - 5)\, 2x}{x^4}}$

(c)

$$h(x) = \tan(e^{\cos x})$$

Chain Rule: $u(x) = e^{\cos(x)}$ $\quad h(x) = h(u(x))$

$\Rightarrow h'(x) = \sec^2(u(x))\, u'(x) = \sec^2(e^{\cos x})\, u'(x)$

But we need the chain rule again to compute $u'(x)$!

$u(x) = e^{\cos x}$ $\quad$ let $v(x) = \cos(x)$, so $u(x) = u(v(x)) = e^{v(x)}$

Then $u'(x) = e^{v(x)}\, v'(x) = e^{\cos(x)}(-\sin x)$

So in total, we have $\boxed{-\sec^2(e^{\cos x})\, e^{\cos x} \sin x = h'(x).}$

2. Evaluate each of the following limits (as a number, ∞, or $-\infty$), or if the limit does not exist, explain why.

(a)

$$\lim_{x\to -3} \frac{2x^2-18}{x+3}$$

Plug in $-3 = x$?

$$\frac{2(-3)^2-18}{-3+3} = \frac{18-18}{3-3} = \frac{0}{0} \quad \text{is inconclusive.}$$

Factoring:

$$\frac{2x^2-18}{x+3} = \frac{2(x^2-9)}{x+3} = \frac{2\cancel{(x+3)}(x-3)}{\cancel{x+3}} = 2(x-3).$$

$$\text{So } \lim_{x\to -3} \frac{2x^2-18}{x+3} = \lim_{x\to -3} 2(x-3) = 2(-3-3) = \boxed{-12}$$

(b)

$$\lim_{x\to\infty} \frac{(2x+1)^3}{-5x^2-4x+3}$$

"Easy" method: The degree of the top is 3, the degree of the bottom is 2. So the top wins, and the limit will be $\pm\infty$. As $x \to \infty$, the top is positive, but the bottom is negative. So the limit is $\boxed{-\infty.}$

Other method:

$$\lim_{x\to\infty} \frac{(2x+1)^3}{-5x^2-4x+3} = \lim_{x\to\infty} \frac{8x^3+12x^2+6x+1}{-5x^2-4x+3} =$$

divide top & bottom by highest power in the denominator

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2}(8x^3+12x^2+6x+1)}{\frac{1}{x^2}(-5x^2-4x+3)} = \lim_{x\to\infty} \frac{8x+12+\cancel{\frac{6}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}}{-5-\cancel{\frac{4}{x}}^{\to 0}+\cancel{\frac{3}{x^2}}^{\to 0}} = \boxed{-\infty}$$

$$\text{or } = \lim_{x\to\infty} \frac{\frac{(2x+1)^3}{x^2}}{\frac{1}{x^2}(-5x^2-4x+3)} = \lim_{x\to\infty} \frac{\frac{(2x+1)^3}{x^2}}{-5+-\cancel{\frac{4}{x}}^{\to 0}+\cancel{\frac{3}{x^2}}^{\to 0}}$$

+ the numerator $\to \infty$

denom $\to -5$

so limit $\to \boxed{-\infty.}$

3. On the curve defined by the following equation, find an expression for dy/dx in terms of x and y.

$$4x^2 + y^3 = 10xy + 2$$

We use implicit differentiation, as we don't have $y = f(x)$, a function of x.

$$\frac{d}{dx}\left(4x^2 + y^3 = 10xy + 2\right) = \frac{d}{dx}4x^2 + \frac{d}{dx}y^3 = \frac{d}{dx}10xy + \frac{d}{dx}2$$

(1) $\frac{d}{dx}4x^2 = 8x$ power rule

(2) $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$ by the chain rule

(3) $\frac{d}{dx}10xy = 10y + 10x\frac{dy}{dx}$ by the product rule

(4) $\frac{d}{dx}2 = 0$

So we get $8x + 3y^2\frac{dy}{dx} = 10y + 10x\frac{dy}{dx}$

$\Rightarrow$ $3y^2\frac{dy}{dx} - 10x\frac{dy}{dx} = 10y - 8x$

$\Rightarrow$ $\frac{dy}{dx}\left(3y^2 - 10x\right) = 10y - 8x$

$\Rightarrow$ $\boxed{\frac{dy}{dx} = \frac{10y - 8x}{3y^2 - 10x}}$

4. Let

$$g(x) = \frac{6x^2 - 6}{x^2 + 4x + 3}.$$

(a) Determine all horizontal asymptotes of the function $y = g(x)$.

$g(x)$ is a rational function.

Horizontal asymptotes concern limiting behavior as $x \to \pm\infty$.

"Easy" method: The degree on top (2) = the degree on bottom (2). So there is a horizontal asymptote, and it's $\boxed{6 = y}$, the ratio of the leading coefficients 6 and 1.

Other method: $\lim_{x \to \infty} \frac{6x^2-6}{x^2+4x+3} = \lim_{x \to \infty} \frac{\frac{1}{x^2}(6x^2-6)}{\frac{1}{x^2}(x^2+4x+3)} = \lim_{x \to \infty} \frac{6 - \cancel{\frac{6}{x^2}}^{\to 0}}{1 + \cancel{\frac{4}{x}}^{\to 0} + \cancel{\frac{3}{x^2}}^{\to 0}} = 6$

(+ the same as $x \to -\infty$). So $\boxed{y = 6}$ is a horizontal asymp.

(b) Determine all vertical asymptotes of the function $y = g(x)$.

Vertical asymptotes happen when the denominator, but not the numerator, is 0.

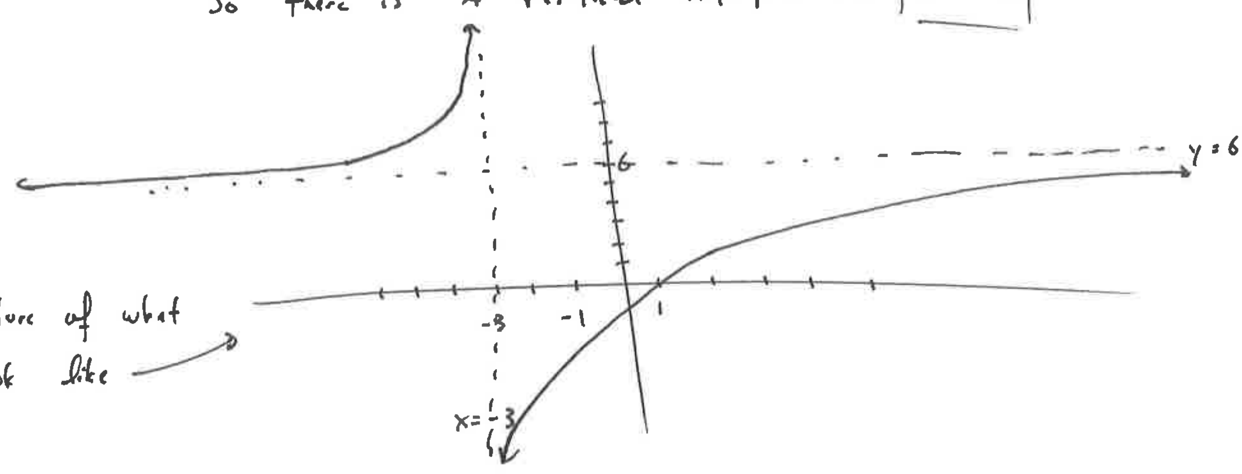
So we factor!

$$6x^2 - 6 = 6(x^2-1) = 6(x+1)(x-1)$$

$$x^2+4x+3 = x^2 + 3x + x + 3 = x(x+3) + 1(x+3) = (x+3)(x+1)$$

So $\frac{6x^2-6}{x^2+4x+3} = \frac{6\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x+3)} = \frac{6(x-1)}{x+3}$

So there is a vertical asymptote at $\boxed{x = -3.}$



A rough picture of what it should look like

5. Let

$$f(x) = x^3 \sin(\pi x).$$

(a) Determine $f'(x)$.

$\frac{d}{dx} \sin(x\pi) = \cos(\pi x)\pi$, where the π came from the derivative of the "inner function" πx

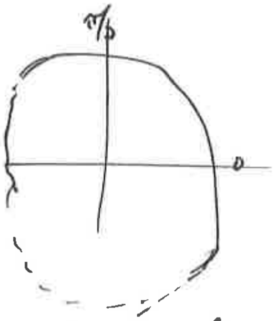
Product + Chain Rules!

$$\boxed{f'(x) = 3x^2 \sin(\pi x) + x^3 \cos(\pi x)\, \pi}$$

(b) Determine the equation of the tangent line to $y = f(x)$ at the point $(1/2, 1/8)$.

$y - y_1 = m(x - x_1)$. $(x_1, y_1) = (\frac{1}{2}, \frac{1}{8})$

The slope m will be $f'(\frac{1}{2}) = 3(\frac{1}{2})^2 \underbrace{\cancel{\sin(\frac{\pi}{2})}}_{1} + \underbrace{\cancel{(\frac{1}{2})^3 \cos(\frac{\pi}{2})}}_{0} \pi$



Bad unit circle

$\sin(\frac{\pi}{2}) = 1$ $\cos(\frac{\pi}{2}) = 0$

So $f'(\frac{1}{2}) = 3(\frac{1}{2})^2 = \frac{3}{4}$

Then the tangent line satisfies $y - \frac{1}{8} = \frac{3}{4}(x - \frac{1}{2})$

or $y = \frac{3}{4}(x - \frac{1}{2}) + \frac{1}{8} = \frac{3}{4}x - \frac{3}{8} + \frac{1}{8} = \frac{3}{4}x - \frac{1}{4}$

So the tangent line is $\boxed{y = \frac{3}{4}x - \frac{1}{4}}$

6. Let $f(x)$ be the function

$$f(x) = \sqrt{3x+1},$$

defined on the domain $x > -1/3$.

(a) Determine $f'(x)$ by using the Chain Rule.

$$f(x) = (3x+1)^{1/2} \quad \text{so} \quad f'(x) = \frac{1}{2}(3x+1)^{-1/2}\,3 = \boxed{\frac{3}{2\sqrt{3x+1}}}$$

or $u(x) = 3x+1$, $\left(f(u(x))\right)' = f'(u(x))\,u'(x) = \frac{1}{2}(u(x))^{-1/2}u'(x)$

$$= \frac{1}{2}(3x+1)^{-1/2}\,3 = \boxed{\frac{3}{2\sqrt{3x+1}}}$$

(b) Determine $f'(x)$ by using the limit definition of the derivative.

Method 1

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} =$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3x+3h+1} - \sqrt{3x+1}\right)\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}{h\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{3x} + 3h \cancel{+1} \cancel{-3x} \cancel{-1}}{h\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)} =$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} = \boxed{\frac{3}{2\sqrt{3x+1}}}$$

Method 2

$$f'(A) = \lim_{z\to A} \frac{f(z) - f(A)}{z-A} =$$

$$= \lim_{z\to A} \frac{\sqrt{3z+1} - \sqrt{3A+1}}{z-A} =$$

$$= \lim_{z\to A} \frac{\left(\sqrt{3z+1} - \sqrt{3A+1}\right)\left(\sqrt{3z+1} + \sqrt{3A+1}\right)}{(z-A)\left(\sqrt{3z+1} + \sqrt{3A+1}\right)}$$

$$= \lim_{z\to A} \frac{3z \cancel{+1} - 3A \cancel{-1}}{(z-A)\left(\sqrt{3z+1} + \sqrt{3A+1}\right)} =$$

$$= \lim_{z\to A} \frac{3\cancel{(z-A)}}{\cancel{(z-A)}\left(\sqrt{3z+1} + \sqrt{3A+1}\right)} =$$

$$= \frac{3}{\sqrt{3A+1} + \sqrt{3A+1}} = \boxed{\frac{3}{2\sqrt{3A+1}}}$$

(I don't know why I used A instead of x → sorry)

7. A particle moves along a line, starting at time $t = 0$, so that its position function at time t is

$$s(t) = 4t^3 - 3t, \ t \geq 0$$

(a) Determine the acceleration of the particle at time $t = 1$.

position : $s(t) = 4t^3 - 3t$

velocity : $s'(t) = 12t^2 - 3$

Acceleration : $s''(t) = 24t$

so acceleration at $t=1$ is $s''(1) = 24(1) = \boxed{24}$

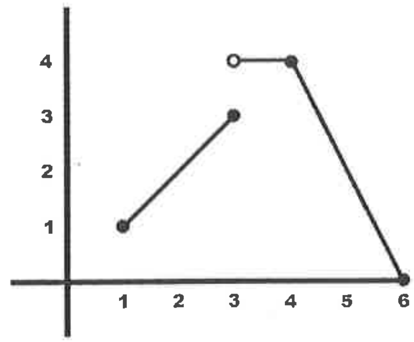
(b) Find all times that the particle is momentarily at rest (not moving).

This is a question about velocity, when $v(t) = s'(t) = 0$.

$s'(t) = 12t^2 - 3 = 3(4t^2 - 1) = 3(2t-1)(2t+1) = 0$

so $s'(t) = 0$ when $\boxed{t = \frac{1}{2}, \quad t = -\frac{1}{2}}$

8. Let $f(x)$ be a function defined on the domain $[1, 6]$, the graph of which is shown in the following image:



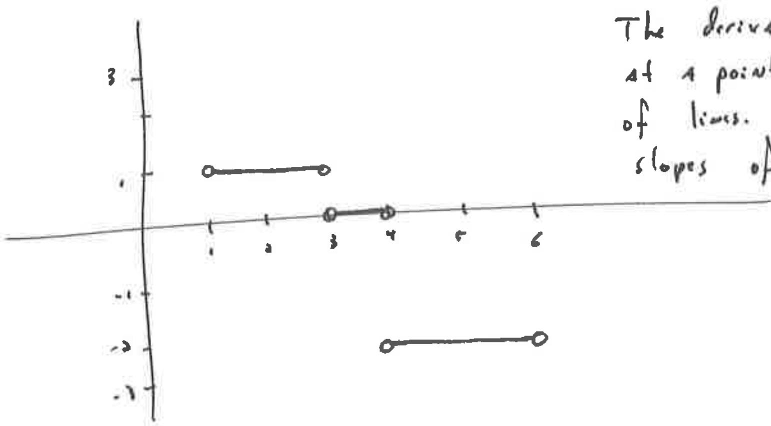
(a) For what values of x (if any) on the interval $(1, 6)$ does $f(x)$ fail to be *continuous*?

f(x) has a jump discontinuity at x = 3.

(b) For what values of x (if any) on the interval $(1, 6)$ does $f(x)$ fail to be *differentiable*?

f(x) is discontinuous ~~~~ at x = 3, so f(x) is not differentiable there. And at x = 4, f(x) has a corner, so f(x) is not differentiable there.

(c) Sketch the graph of $y = f'(x)$.



The derivative gives the slope at a point. f(x) is a progression of lines. So we graph the ~~~~ slopes of the three lines.